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MATHEMATICAL MODELS OF THE PROBLEM OF CONSTRUCTING DELIVERY ROUTES OF CARGO IN THE INTERNAL ZONES OF TRUNK NODES OF A HIERARCHICAL TRANSPORT NETWORK

The paper formulates a mathematical model of the problem of constructing circular routes for the delivery of discrete cargo in the transport network. The types of costs of real transport processes, which should be taken into account in the formation of the objective function of the routing problem with a heterogeneous fleet of vehicles, are discussed. The possibility of solving the formulated problem with the help of well-known packages of mixed and integer linear programming is noted.

Keywords: *problems of combinatorial optimization, mathematical models of circular routes of vehicles.*

Introduction. Most of the well-known works devoted to the solution of routing problems in transport consider idealized (academic) mathematical models that do not take into account many factors inherent in the real processes of cargo (cargo) handling and transportation. For example, such important parameters as distance, travel time, transportation costs, service time, and others are present in some abstract form. In many models, the Euclidean distance between the nodes of the transportation network is assumed when the triangle rule is satisfied. Speed and cost, with a few exceptions from stochastic models, are modeled by constants. In real life, the actual distances, time and present costs of transporting cargo must be calculated for transportation. In particular, in urban areas representing the transport network, travel times are very uncertain, as the speed of vehicles can change over time due to congestion on the streets. Therefore, it seems reasonable to use the average speed of traffic to calculate the delivery time of cargo, but it should be taken into account that with an increase in the average speed on bypass roads, operating costs may increase. In any case, in mathematical models of transportation, it is necessary to take into account the relationship between speed and cost. In addition, when calculating the delivery time, customers need to take into account the time for sorting and loading cargo at the central hub and the time for unloading and loading cargo at the locations of customers.

Problems in which the working fleet of vehicles is homogeneous are often considered. In practice, as a rule, transport companies have a heterogeneous fleet of vehicles. There are other important operational features in addition to the heterogeneity of the fleet, for example, the routing problem must be considered in conjunction with the problem of optimizing the distribution of drivers between vehicles, taking into account the constraints on working hours. In the classic formulation of the routing problem, there is a single depot where each vehicle starts and ends its movement.

At the same time, only one cyclic route (tour) of movement is allowed for each vehicle. In real-world cargo transportation problems, there may be multiple depots, and drivers can perform many different tours per day.

There are several main classes of transport routing problems, which are most often considered in the literature:

1) Classic Vehicle Routing Problem (The Vehicle Routing Problem, VRP) and its varieties: Simple Pickup – Delivery (Vehicle Routing Problem with Pickup and Delivery, VRPPD); Delivery with the ordering of loaded cargo according to the rule "last in, first out" (VRPPD with LIFO); Delivery with time windows (Vehicle Routing Problem with Time Windows, VRPTW); Delivery with limited vehicle capacity on routes or with limited capacity of network arcs (Capacitated Vehicle Routing Problem, CVRP or Capacitated Arc Routing Problem, CARP); Delivery with simultaneous assembly of cargo for the depot (Vehicle Routing Problem with Simultaneous Delivery and Pick-up, VRPSDP); Delivery with repeated visits to customers (Split Delivery Vehicle Routing Problem, SDVRP); delivery, where the same vehicles can be used repeatedly on different routes (Vehicle Routing Problem with Multiple Trips, VRPMT); delivery, when vehicles can have more than one depot (Vehicle Routing Problem with Multiple Depots, VRPMD); An open routing task where vehicles are not required to return to the depot (Open Vehicle Routing Problem, OVRP); Truck and trailer routing task (The Truck and Trailer Routing Problem, TTRP); the task of routing vehicles with trailers, when trailers can remain in nodes for unloading and loading (The Rollon-Rolloff Vehicle Routing Problem, RRVRP);

2) A generalized problem of formation (determination of size and composition) of a heterogeneous working fleet of vehicles and a scheme for the distribution of cargo flows and routing of vehicles (Heterogeneous Fleet Vehicle Routing Problem, HFVRP or The Fleet Size and Mix Vehicle Routing Problem, FSMVRP) and its varieties depending on the constraints on the size of the park; A generalized problem of forming a heterogeneous working fleet of vehicles, locating a depot and distributing cargo flows and routing vehicles (The Fleet Size and Mix Location Routing Problem, FSMLRP); The Network Design Challenge (Network Design Problems, NDP) and the task of network maintenance (Service Network Design problems, SND) in which routing is related to the frequency, modes and schedules of service work.

The second class of problems is sometimes generalized under the name Fleet Composition and Routing Problems, FCRP. In practice, all these tasks can be interconnected and form different hybrids, depending on the need to take into account the actual constraints. The considered classes of tasks can be divided and arranged according to several fundamental features. For example, the number of depots and customers, the technical characteristics of the vehicles and the physical characteristics of the road network and the cargo transported (continuous and discrete flows, homogeneous and mixed cargoes), schemes and conditions of cargo transportation, time constraints, etc.

The VRP problem was first formulated in 1959 by G.B. Dantzig and J.H. Ramser and later became known as the classical problem. The NP difficulty of the classical VRP problem was proved in 1981 by J.K. Lenstra and A.H.G. Rinnooy Kan [1]. For this reason, it is now generally accepted that for most varieties of the VRP problem and the general TSP problem, there is no Polynomial and Fully Polynomial Time Approximation scheme (PTAS and FPTAS), unless NP≠P. In 1964, G. Clarke and J. Wright were the first to propose an approximate algorithm for solving the VRP problem. In the following years, intensive research began on methods and algorithms for solving this problem and its varieties [2].

In contrast to the classical VRP problem and its variants, the problems of the FCRP family have been studied less intensively. These problems consider not only transport costs, but also the costs associated with the acquisition of a heterogeneous working fleet of vehicles. In foreign literature, it is customary to divide problems with a heterogeneous fleet depending on the limited or unlimited number of vehicles of each type, taking into account the fixed cost of vehicles and the variable cost of transporting cargo, as well as on whether transportation prices depend on the type of vehicles. According to the foreign classification, there are five main classes of FCRP problems: Heterogeneous VRP with Fixed Costs and Vehicle Dependent

Routing Costs (HVRPFD) – a limited fleet, fixed and variable costs are taken into account, prices depend on the type of vehicle; Heterogeneous VRP with Vehicle Dependent Routing Costs (HVRPD) – limited fleet, only variable cost is taken into account, prices depend on the type of vehicle; Fleet Size and Mix VRP with Fixed Costs and Vehicle Dependent Routing Costs (FSMFD) – unlimited fleet, fixed and variable costs are taken into account, prices depend on the type of vehicle; Fleet Size and Mix VRP with Fixed Costs (FSMF) – unlimited fleet, fixed and variable costs are taken into account, prices do not depend on the type of vehicle; Fleet Size and Mix VRP with Vehicle Dependent Routing Costs (FSMD) – unlimited fleet, only variable cost is taken into account, prices depend on the type of vehicle.

The first studies of FCRP problems are considered to be the works of G. B. Dantzig, D. R. Fulkerson (1954) [3] and D. Kirby (1959) [4]. In 1984, a review article was published by B. L. Golden, A. Assad et al. [5], which discusses possible models and methods of operations research applicable to solving FCRP problems. In particular, this paper discusses the Fleet Size and Mix (FSM) problem, which determines the optimal composition and routing scheme of vehicles. To solve the problem, the authors proposed two heuristic algorithms based on the savings algorithm of Clark and Wright (G. Clarke and J. Wright) [6] and the construction of a large traveling salesman tour. They also formulated a mathematical model for the FSMF problem and presented some lower bounds on its solution.

A review of early work on FCRP problems was given by S. Salhi and G.K. Rand in 1993 [7]. I.H. Osman and S. Salhi [8] researched and analyzed works up to 1996 devoted to solving problems using local search methods. Y.H. Lee et al. (2008) [9] describe several heuristic approaches to solving various variants of FCRP problems based on tabu search and set partitioning methods. More recent reviews of the problems of forming a heterogeneous fleet of vehicles are given by R. Baldacci et al. [10, 11] and A. Hoff et al. [12, 13]. Some of the latest works by R. Baldacci et al. [14–17] are devoted to the study of the issues of improving lower bounds, the review of exact methods, and the comparison of the computational efficiency of algorithms for solving VRP and FCRP problems. In [15] proposes one of the currently best accurate algorithms based on splitting sets and allows finding optimal solutions to FCRP problems with up to 100 clients. The algorithm uses three types of Lagrangian and LP relaxation procedures of the initial mathematical formulation of the problem, which allow you to significantly reduce the number of variables of the problem and use well-known packages of integer linear programming programs to solve it. The article presents the results of calculations for test instances of FCRP problems, which showed new better lower bounds of the solution for individual problems, and that the proposed precise algorithm for the first time made it possible to solve several previously unsolved test cases.

1. Cost Functions and Meaningful Problem Statement

This half section discusses issues related to the methodology for calculating the present costs in solving the problem of routing traffic for long-term and medium-term periods of time. The purpose of solving this problem is to determine the total number and composition of vehicles by type and carrying capacity necessary to perform all transportation in the inner zone of the central trunk nodes of the global hierarchical network while minimizing capital costs for the purchase of vehicles and operating costs for the transportation of cargo. When solving the problem, it is necessary to determine the working fleet of heterogeneous vehicles (heterogeneous working fleet) and find a scheme for the distribution of cargo flows and the routing of vehicles in the intra-node transportation network. In foreign literature, these tasks are commonly referred to as Fleet Size and Mix Vehicle Routing (FSMVRP) and Heterogeneous Fleet Vehicle Routing Problem (HFVRP), depending on the constraints on the number of vehicles each.

For transport enterprises (companies), the key factors in calculating costs are the expected volumes of transportation, the prices of vehicles and the cost of transportation. Vehicles with a higher payload capacity generally have a lower unit cost than vehicles with a lower payload, provided that the vehicle load factor is high enough. It should also be taken into account that for used vehicles, depreciation costs are lower and maintenance costs are higher than for new ones. Traffic volumes and vehicle prices change over time, so transport companies also have the problem of managing the size of the working fleet not only for the current planning periods, but also for the future. An excessively large fleet with a decrease in demand for transportation forces the sale or lease of surplus vehicles, an increase in demand leads to the need to purchase new vehicles or lease them. In any case, in long-term planning, the expected revenues should be greater than the expected costs. Regulation of the size of the working fleet, depending on the above and other random factors, should be carried out when solving dynamic stochastic problems of long-term planning, the mathematical models of which are more aggregated than those of current planning problems. Accordingly, in this case, more aggregated reduced cost functions are used.

Mathematical models of current (tactical) planning, as a rule, consider average flows, and prices for given periods of time throughout the year, calculated by statistical methods. In this case, the uncertainty is reduced, and the routing problem should be considered at a more detailed level, taking into account all relevant factors. As in long-term planning, in the face of fluctuating demand, the main decisions are related to the purchase and lease of vehicles or the sale and lease of existing surpluses. However, when making tactical decisions, more attention should be paid to adjusting the carrying capacity of vehicles. Therefore, when solving routing problems for medium-term periods, as a rule, reserves of the carrying capacity of vehicles are created on all routes of cargo transportation, i.e. the maximum load factor of the vehicle can be set. This leads to the use of vehicles with a higher load capacity and an increase in capital and operating costs, but allows customer orders to be fulfilled in the face of fluctuating demand within certain constraints. The challenge is to define a "middle ground". In the current planning, the size and composition of the working fleet of vehicles, a detailed scheme of the distribution of cargo and vehicle flows along the routes are precisely known. Concluded contracts and tariffs for the transportation of cargo and other types of operations that affect the current financial condition of the transport company are also known. In this regard, with uncertain fluctuations in demand, the task of optimizing the working fleet of vehicles and the portfolio of orders can be set – it is necessary to select the most profitable contracts so that they can be fulfilled with the existing potential of the transport company's carrying capacity. It should be noted that it is not always necessary to combine the solution of the problem of determining the working fleet of vehicles with the solution of the problem of managing the assets and portfolio of orders of the transport company, since the latter can be solved independently under known fluctuations in the loads in the transportation network.

At the operational level, the task of the transport company is usually to redistribute flows and reoptimize the routing scheme in the event of exceeding capacity or the occurrence of failures for customers and routes, as well as in various unforeseen situations and natural disasters. In this case, the internal reserve of vehicles can be used or rental vehicles can be involved.

Herewith for transport companies, it is extremely important that they be able to obtain realistic estimates of transport costs in their current planning tasks. Therefore, in the mathematical models used for current transport planning, the cost functions should reflect the main operating costs associated with the maintenance of the fleet of vehicles and the transport of cargo.

In the majority of foreign and domestic works devoted to solving the problems of closed routing (building cyclic routes), the authors consider the costs on the route arcs as given constants, and assume that these costs do not depend on the type and carrying capacity of the vehicle. Moreover, it is also assumed that the costs are the same for the passage of the transport arc in the forward and reverse directions. However, such assumptions are not realistic. In practice, the cost of curves depends on the length, condition and geographical features of the road sections, the type and carrying capacity of the vehicle, its current load and speed. All this, in turn, affects fuel consumption. In addition, when transporting cargo, it is necessary to take into account additional costs associated with accidental factors – forced downtime of vehicles due to unforeseen situations on the road, natural disasters, etc.

In circular routing problems, the start and end points of the route are always known, so all the nodes and arcs through which the route passes will be found. This makes it possible, provided that the costs are additive, to specify them on the arcs (i, j) and in nodes i functions f_{ij}^k and f_i^k for each vehicle type k. In addition to the basic parameters determining the length of the arc, the carrying capacity and the speed of the vehicle, these functions shall include factors reflecting the actual load of the vehicle on the arcs and her geographical features (e.g. the ascent or descent of the relevant section of road). Functions f_{ij}^k can also depend on a single, generalized parameter – fuel consumption on the arc (i, j). The values of these functions calculated in the process of solving the problem should determine the real operating costs for the transportation of cargo along the entire route passing through physical sections of roads.

Figure 1 illustrates fragments of the internal zones of the central trunk nodes of the global hierarchical network, schematically shows the circulating flows and an example of circular routes, where: *a*) a network with a central node (depot), ■ – locations of customers, ◆ – road junctions, road sections are shown by lines; *b*) incoming and outgoing depot flows (wide arrow), and nodal flows to and from customers (simple arrow); *c*) two circular routes of vehicles.

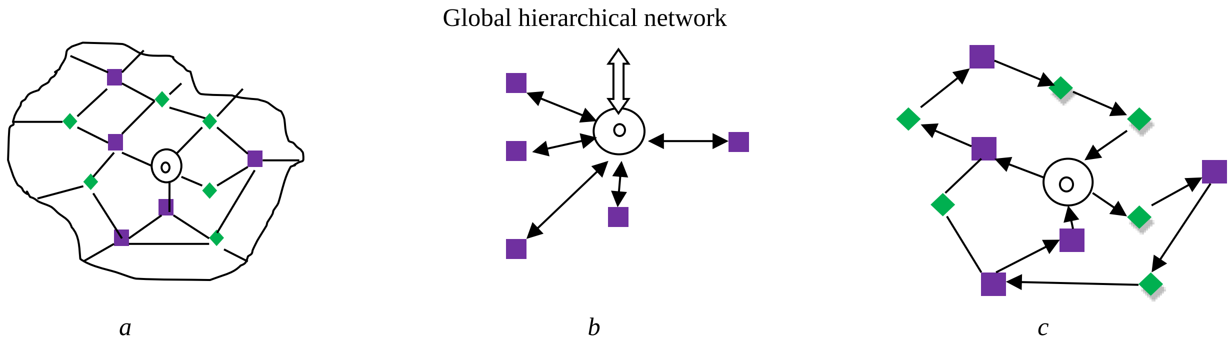

FIG. 1. Fragments of an intra-node network: *a* – a network with a central node; *b* – depot flows and nodal flows; *c* – two circular routes of vehicles

It is clear that the costs of acquiring, maintaining and operating a working fleet of vehicles in solving the routing problem should be calculated only taking into account those parameters that are explicitly or implicitly included in the mathematical model of the problem. Therefore, these costs will be only a part of the cost of transportation of the transport company. Recall that the cost of transportation is the value of operating costs of a transport enterprise, expressed in monetary form, falling on average per unit of transport production. In road freight transport, the cost of transportation is determined, as a rule, per ton of cargo transported per kilometer (t/km) or per use of one vehicle per hour (a/h). Cost calculation is necessary to determine the tariffs for the transportation of cargo and the expected profit of the enterprise. Optimization of cargo transportation routes makes it possible to reduce the cost of transportation mainly by eliminating irrational routes, increasing the load factor and reducing the consumed fuel and empty mileage of vehicles, reducing the time of delivery of cargo to the consumer.

Since the capital costs for the acquisition of a working fleet of vehicles are known after solving the problem, the question arises: which items of operating costs for transportation should be taken into account in the general formula for calculating the present costs when solving the problem? It seems reasonable that in order to calculate operating costs, the formula should take into account only those cost items that are directly related to the transportation of cargo, and the general business (overhead) costs of the enterprise can be ignored. Let's list the main items of these costs: fuel, lubricants and other operating materials; maintenance and repair of rolling stock (cars and trailers); depreciation for the restoration of rolling stock; wear and repair of automotive tires; drivers' salaries together with social contributions. In addition to the above, other items of operating costs related to traffic (tolls, toll roads, etc.) can also be taken into account in the calculated general formula of the present costs. The main thing is that the value of operating costs, calculated according to the derived formula for solving the problem, should be as close as possible to be comparable (adequate) with the value of that part of the costs that were actually incurred by the enterprise only for the transportation of cargo along the optimized routes of vehicles. The determination of the general formula for calculating the present costs is a separate, difficult problem for the economists of a transport enterprise, which must be solved before the numerical solution of the problem of finding the composition and number of vehicles, distributing cargo flows and routing vehicles is carried out. In many practical cases, the present cost functions are nonlinear, non-separable, and non-additive, and this should be taken into account when constructing a mathematical model of the problem under consideration.

2. Mathematical Model of the Network of Transportation

Let the physical network of intra-junction transportation (see Fig. 1,*a*) be given by a connected directed graph $\tilde{G}(\tilde{N},\tilde{P})$. Set of vertices $\tilde{N}=\{0\}\cup\{1,...,n\}\cup\{n+1,...,\tilde{n}\}$, renumbered from 0 to $\tilde{n}$. Includes Vertex $\{0\}$ – depot and central hub, $\{1,...,n\}$ – set of clients and $\{n+1,...,\tilde{n}\}$ – a set of transit points connecting individual sections of roads. Transit points are introduced to take into account the characteristics of individual sections of roads connecting nodes in the inner zone of the central node. Road sections are represented by oppositely directed oriented arcs (i,j), (j,i), $i,j\in\tilde{N}$, $i\neq j$, one of which may be absent (for example, due to one-way traffic on a section of the road or a ban on the passage of freight transport). The lengths of the arcs are given by the matrix $\tilde{R}=\|\tilde{r}_{ij}\|_{\tilde{n}+1\times\tilde{n}+1}$, $\tilde{r}_{ij}\in R^1$, $(i,j)\in\tilde{P}$.

Let the coefficients for the network arcs be known $\tilde{k}_{ij}\in R^1$, $\tilde{k}_{ij}\geq 1.0$, characterizing the geographical features (ascent, descent, etc.) and physical condition (quality of pavement) of road sections. Let's define the elements of the transformed matrix $\tilde{R}'$ how $\tilde{r}'_{ij}=\tilde{k}_{ij}\tilde{r}_{ij}$, $(i,j)\in\tilde{P}$, and build for it the shortest paths to $\tilde{G}(\tilde{N},\tilde{P})$. As a result, we get a matrix of lengths of the shortest paths $\tilde{L}=\|\tilde{l}_{ij}\|_{\tilde{n}+1\times\tilde{n}+1}$ and reference matrix $\tilde{C}=\|\tilde{c}_{ij}\|_{\tilde{n}+1\times\tilde{n}+1}$, each element of which $\tilde{c}_{ij}$, $i\neq j$ determines the number of the penultimate node on the shortest path from i before j, $\tilde{c}_{ii}=0$, $i=\overline{1,\tilde{n}+1}$. With the help of a reference matrix, you can easily determine the shortest path between any vertices of the original graph. By matrix $\tilde{L}$ For the set $\{0\}\cup\{1,...,n\}$ Let's construct a complete directed graph $G(N,A)$ with a set of vertices $N=\{0,1,...,n\}$ and a set of arcs $A=\{(i,j):\forall i,j\in N, i\neq j\}$ with known lengths l_{ij}, $(i,j)\in A$. Obviously, for the arcs of the constructed graph, the triangle rule holds $l_{ik}+l_{kj}\geq l_{ij}$. In the future, all problems will be formulated on the graph $G(N,A)$.

Let each of the clients j, $j=\overline{1,n}$, Set average daily quantity $a_j>0$ and $b_j>0$, $a_j,b_j\in Z^+$, units of discrete cargo of a uniform size, which must be delivered to the client from the central hub (depot) within a day and sent from the client to the central hub ($a_0=b_0=0$). It is assumed that during a certain period of current planning of the duration of T the average daily fluxes change slightly and according to the T days can be transported $T\sum_{j=1}^{n}(a_j+b_j)$ cargo.

The depot has K types of vehicles with different carrying capacities $Q_k\in Z^+$, $k=\overline{1,K}$. Load capacity is measured in the same units as discrete cargo flows. It is assumed that $\max_j\{a_j,b_j\}\leq Q_K$, and the number of vehicles of each type may be limited by the size of the m_k, $k=\overline{1,K}$ And it's not limited. For each type of vehicle, the cost of its purchase and the average cost of maintenance per day are known. The cost of maintenance includes the cost of lubricants and other operating materials, repair and depreciation of rolling stock, wear and repair of automobile tires, drivers' salaries along with deductions for social needs. Let the function be given to calculate these costs $F_V = f(S_k, R_k)$, $k = \overline{1,K}$, where S_k – fixed cost of purchasing a type vehicle k, and R_k – maintenance costs of one type vehicle k per day.

After sorting at the central hub, the cargo for each customer can be transported in vehicles on pallets, for loading and unloading of which it is necessary to use mechanized forklifts. Cargo from customers to the central hub are transported "in bulk". With a large volume of transported cargo, the costs of loading and unloading operations can be significant and must be taken into account when solving the problem. For the depot and customers, the function of loading and unloading costs is specified u_j cargo $F_L = f(Q_k, u_j)$, $j = \overline{0,n}$, $k = \overline{1,K}$. In practice, it is generally accepted that this function does not depend on the type of vehicle, but depends only on the volume of cargo and the availability of loading and unloading equipment in the depot and at customers. In the depot, the costs of loading and unloading are borne by the transport company, customers can perform these operations at their own expense. In the future, it is assumed that the locations of customers are the points of delivery and collection of the transport company, so the costs of loading and unloading cargo from customers also belong to the total costs of the enterprise. As a rule, such costs are modeled by a continuous concave function $F_L = f(u_j)$, $j = \overline{0,n}$ of the volume of cargo. In some cases, a linear relationship can be adopted $F_L = c_L u_j$, where c_L – the cost of loading and unloading a unit of discrete cargo. It is clear that if the costs of loading and unloading cargo do not depend on the type of vehicle, then they should not appear in the objective function of the problem, and can be calculated independently of the variables as follows: $F_{full} = f(\sum_{i=1}^{n}(a_i + b_i)) + \sum_{i=1}^{n} f(a_i + b_i)$ or $F_{full} = 2c_L \sum_{i=1}^{n}(a_i + b_i)$. If such a relationship exists, then these costs should be related to the unknown variables of the problem.

Let the fare of a vehicle type k in an arc $(i, j) \in A$ given by the function $F_M = f(l_{ij}, Q_k, V_{av}^k, k_L)$, where l_{ij} – the length of the arc in kilometers, V_{av}^k – average speed (km/h), k_L – vehicle load factor on the arc. Meaning $k_L = u_{ij} / Q_k \in [0,1]$, where u_{ij} – current loading of the vehicle. Essentially, the F_M determines the cost of fuel consumed for arc movement (i, j) for each vehicle type k. In many routing tasks, $F_M = c_k l_{ij}$, where c_k – specific fuel cost of vehicle type k for one kilometer of travel. It is often considered that the costs of moving in an arc do not depend on the type of vehicle and are the same for the forward and reverse direction of travel. In this case, additional conditions for carrying capacity are introduced $Q_1 < Q_2 ... < Q_K$ and fixed cost $F_1 < F_2 ... < F_K$.

All cost functions must be reduced to a comparative form, for example, per day or for a given period of time of current planning T. If the cost functions are built F_V, F_L and F_M realistically reflect production costs, their amount will be sufficiently close to the actual costs of the transport company for the purchase and operation of the working fleet of vehicles, excluding overhead costs. Let us assume that when solving the problem, these functions and their corresponding numerical values are calculated F_k, f_j^k and f_{ij}^k, $i, j = \overline{0,n}$, $k = \overline{1,K}$ are used in the objective function.

3. Mathematical model of the problem of constructing delivery routes for the transportation of cargo from the depot to customers. Option 1

The formulation of the problem is based on the models proposed by Gheysens et al. [18], Golden et al [5], Baldacci et al [10] and Salhi and Rand [7]. The task is to determine the set of Hamiltonian cyclic routes with minimum total costs that do not intersect among customers and begin and end in the depot. At the same time, it is necessary that all customer requests are satisfied, and the carrying capacity of vehicles on all routes is not violated.

Let's denote $C=\{1,...,n\}$ – a set of clients, $V=\{1,...,K\}$ – multiple types of vehicles. Let's introduce streaming variables y_{ij}, that determine the amount of cargo in the vehicle when it passes to the client j after the customer's visit i, $i, j \in N$, and Boolean variables x_{ij}^k, $x_{ij}^k = 1$ if the vehicle is of the type k moves away from the customer i to the client j and $x_{ij}^k = 0$ otherwise. Let $\sum_{j=1}^{n} x_{0j}^k$ is the total number of vehicles of the type used k. Need to find a minimum of function

$$F_{DR} = \sum_{k\in V} F_k \sum_{j\in C} x_{0j}^k + \sum_{k\in V}\sum_{j\in N} f_j^k \Big(\sum_{i\in C} \big|(y_{ij} - y_{ji})\big| x_{ji}^k\Big) + \sum_{k\in V}\sum_{i,j\in N} f_{ij}^k x_{ij}^k, \tag{1}$$

subject to

$$\sum_{k\in V}\sum_{i\in N} x_{ij}^k = 1, \quad \forall j \in C, \tag{2}$$

$$\sum_{i\in N} x_{ij}^k - \sum_{i\in N} x_{ji}^k = 0, \quad \forall j \in C, \quad \forall k \in V, \tag{3}$$

$$\sum_{j\in C} y_{0j} = \sum_{j\in C} a_j, \quad \sum_{j\in C} y_{j0} = 0, \tag{4}$$

$$\sum_{i\in N} y_{ij} - \sum_{i\in N} y_{ji} = a_j, \quad \forall j \in C, \tag{5}$$

$$y_{0j} \le \sum_{k\in V} Q_k x_{0j}^k, \quad \forall j \in C, \tag{6}$$

$$y_{ij} \le \sum_{k\in V} (Q_k - a_i) x_{ij}^k, \quad \forall i \in C, \quad \forall j \in N, \quad i \ne j, \tag{7}$$

$$x_{ij}^k \in \{0,1\}, \quad y_{ij} \ge 0 \text{ and integer}, \forall i, j \in N, \quad \forall k \in V. \tag{8}$$

Constraints (7) can be written in extended form

$$a_j x_{ij}^k \le y_{ij} \le (Q_k - a_i) x_{ij}^k, \ \forall i \in C, \ \forall j \in N, \ i \ne j, \ \forall k \in V. \tag{9}$$

If the number of vehicles of type k limited by the magnitude m_k, then conditions can be added to the problem

$$\sum_{j\in C} x_{0j}^k \le m_k, \quad \forall k \in V. \tag{10}$$

If necessary, time constraints can be set on the maximum duration of routes. To do this, let's include time parameters in the model T_k – maximum travel time of a vehicle type k and t_{ij} – arc travel time (i, j), as well as continuous variables r_{ij}, that determine the difference between the value T_k and the current value of the driving time after driving in an arc (i, j). Then constraints will be added to the problem

$$r_{ij} \le \sum_{k \in V} T_k x_{ij}^k, \quad \forall (i,j) \in A, \tag{11}$$

$$r_{0j} = \sum_{k \in V} T_k x_{0j}^k - \sum_{k \in V} t_{0j} x_{0j}^k, \quad \forall j \in C, \tag{12}$$

$$\sum_{i \in N} r_{ip} - \sum_{j \in N} r_{pj} = \sum_{k \in V} \sum_{j \in N} t_{pj} x_{pj}^k, \quad \forall p \in C, \tag{13}$$

$$r_{ij} \ge 0, \quad \forall (i,j) \in A. \tag{14}$$

The first part of the objective function (1) determines the fixed costs for the purchase and maintenance of the vehicles used, and the second and third – the costs of loading, unloading and delivery of all cargo. Constraints (2) and (3) ensure that each customer j visited only once by any vehicle of the type k, and this vehicle after arriving at the client and unloading his cargo a_j must leave the client. Constraints (4) mean that the total number of cargoes at the depot exit is equal to the total requirements of all customers, and that consignments are not returned to the depot. Constraints (5) mean that with a customer visit j cargo a_j must be unloaded, i.e. the amount of cargo in the vehicle after visiting the customer j decreases by an amount a_j. These constraints ensure that the requirements of all customers are met and eliminate cycles that do not go through the depot. Constraints (6) determine that the number of loads y_{0j}, transported to the customer j, must not exceed the carrying capacity of the vehicle intended for the delivery of this customer's cargo. Constraints (7) or (9) bind variables y_{ij} and x_{ij}^k, and mean that no cargoes are transported from the i into j, if no vehicle maintains communication between these nodes, i.e. if $x_{ij}^k = 0 \ \ \forall k \in V$. Constraints (8) establish the scope of variables. Constraints (11) mean that the remaining driving time of the vehicle after passing in an arc (i, j) may not exceed the maximum driving time of the vehicle. Conditions (12) ensure that the driving time of the vehicle remaining after leaving the depot is equal to the difference between the maximum travel time and the time required to move to the customer j. Constraints (13) indicate the fact that each time a vehicle passes between two customers; the driving time is reduced by the travel time between those customers.

To formulate the problem (1)–(8) with time window constraints, the depot is represented by two nodes with numbers 0 and $n+1$. Then $N = \{0\} \cup \{1,...,n\} \cup \{n+1\}$. Let $[t_i^S, t_i^C]$ and s_i determine the period of time (time window) during which service is possible, and the time of customer service i, $t_i^S, t_i^C, s_i \in R^1$. Let's introduce variables $t_i^k \in R^1$, what do the exact start time of service of the type vehicle k client's i. It is accepted that the vehicle arrives at the customer's location i previously t_i^S, and it waits for service to begin. To the problem (1) - (8) will be added – to the objective function of the term $\sum_{k \in V} (t_{n+1}^k - t_0^k)$ and limitations [19]:

$$\sum_{j \in N} x_{0j}^k = 1, \quad \sum_{i \in N} x_{i,n+1}^k = 1, \quad \forall k \in V;$$

$$t_j^k \ge t_i^k + s_i + t_{ij} - M_{ij}(1 - x_{ij}^k), \quad \forall i, j \in N, \quad \forall k \in V,$$

where $M_{ij} = \max\{0, t_i^C + s_i + t_{ij} - t_j^S\}$; $t_i^S \sum_{j \in N} x_{ij}^k \le t_i^k$, $t_i^k \le t_i^C \sum_{j \in N} x_{ij}^k$, $\forall i \in N$, $\forall k \in V$; $x_{i0}^k = 0$, $x_{ii}^k = 0$, $x_{n+1,i}^k = 0$, $\forall i \in N$, $\forall k \in V$.

In the record of the first constraints, it is assumed that if the vehicle is not in use, then it makes an empty tour, i.e. $x_{0,n+1}^k = 1$, $\forall k \in V$. Besides $a_0 = a_{n+1} = s_0 = s_{n+1} = 0$, $[t_0^S, t_0^C] = [t_{n+1}^S, t_{n+1}^C] = [Start, Close]$, where *Start* i *Close* – time of departure and arrival at the depot. A time window problem is a mixed integer programming problem.

4. Mathematical model of the problem of constructing delivery routes for the transportation of cargo from the depot to customers. Option 2

Let's enter instead of variables y_{ij} Boolean variables $y_{ik} = 1$, if the vehicle is of the type k visits the customer or depot i, and $y_{ik} = 0$ otherwise. To exclude subcycles when building routes, we introduce integer variables $u_{ik} \ge 0$, whose values determine the customer's sequence number i in the route of a vehicle of the type k. For the first time, this option for excluding subcycles was proposed in 1960 in a paper [20].

Let's formulate the problem. It is necessary to find the minimum function

$$F_{DR} = \sum_{k \in V} F_k \sum_{j \in C} x_{0j}^k + \sum_{k \in V} f_0^k \left(\sum_{i \in C} a_i y_{0k}\right) + \sum_{k \in V} \sum_{i \in C} f_i^k (a_i y_{ik}) + \sum_{k \in V} \sum_{i,j \in N} f_{ij}^k x_{ij}^k \tag{15}$$

subject to

$$\sum_{k \in V} y_{ik} = 1, \ \forall i \in C, \tag{16}$$

$$\sum_{j \in N} x_{ij}^k = y_{ik}, \ \sum_{j \in N} x_{ji}^k = y_{ik}, \ \forall i \in N, \ i \ne j, \ \forall k \in V, \tag{17}$$

$$\sum_{i \in C} a_i y_{ik} \le Q_k, \ \forall k \in V, \tag{18}$$

$$u_{ik} - u_{jk} + n x_{ij}^k \le n - 1, \ \forall i, j \in C, \ \forall k \in V, \tag{19}$$

$$x_{ij}^k, y_{ik} \in \{0,1\}, \ \forall i, j \in N, \ \forall k \in V, \tag{20}$$

$$u_{ij} \ge 0 \text{ and integer}, \ \forall i, j \in C. \tag{21}$$

Note that the conditions for prohibiting subcycles can be written as

$$\sum_{i \in S} \sum_{j \in S \setminus \{i\}} x_{ij}^k \le |S| - 1, \ \forall S \subseteq C, \ |S| \ge 2, \ \forall k \in V, \tag{22}$$

where S – cluster (subset of clients), which is served by one transport service of the same type k.

Constraints (16) and (17) ensure that each customer is visited by only one vehicle, and that the vehicle must leave it after visiting the customer or depot. Conditions (18) and (19) or (22) set constraints on vehicle loading and prohibit sub-cycles not passing through the depot. You can also enter additional conditions into the task that limit the number of vehicles of each type and time constraints on routes.

5. Mathematical model of the problem of constructing delivery routes for the transportation of cargo from the depot to customers. Option 3

Let us consider the two-product formulation of the problem proposed in [21], for the case when $f_{ij}^{k_1} = f_{ji}^{k_2} = f_{ij} = f_{ji}$ $\forall i, j \in N$, $k_1 \neq k_2 \in V$, $f_i^k = f_i$ $\forall i \in N$, $\forall k \in V$. Loading and unloading costs are not included in the model, as it is assumed that they do not depend on the type of vehicle and can be calculated using the formula F_{full}.

Let $G(N', E)$ – A complete undirected graph derived from a graph $G(N, A)$ As follows. Set of nodes N' includes a set of client nodes, K copies of the depot node associated with K vehicle types and one common depot hub. Let the nodes N' numbered and $N' = \{1,...,n\} \cup \{n+1,...,n+K\} \cup \{n+K+1\}$. Let's designate, as before, $C = \{1,...,n\}$, $V = \{n+1,...,n+K\}$, $n' = n+K+1$. Let $\pi(i)$, $i \in V$ means a vehicle associated with a node i. Let's determine the cost of arcs $c_{ij} \in E$ as follows: $c_{ij} = F_{\pi(j)} + f_{0i}$, for $a_i \le Q_{\pi(j)}$, $j \in V$, $i \in C$; $c_{ij} = f_{ij}$, for $a_i + a_j \le Q_K$, $i, j \in C$, $i < j$; $c_{in'} = f_{0i}$, for $i \in C$; $c_{ij} = +\infty$ otherwise. It is assumed that $a_i = 0$, $\forall i \in V \cup \{n'\}$.

The model uses two flow variables y_{ij} and y_{ji}, associated with each arc $(i, j) \in E$. Variable y_{ij} determines the load of the vehicle, and the variable $y_{ji} = Q_K - y_{ij}$ is the free capacity of the vehicle with the highest carrying capacity. Meaning $y_{ji} - (Q_K - Q_k)$ means the free capacity of a vehicle of type k. Let each arc $(i, j) \in E$. Boolean variables defined $x_{ij} = 1$, if the arc (i, j) included in the solution and $x_{ij} = 0$ otherwise.

Significant $\Omega = \{S : S \subseteq C, |S| \ge 2\}$ and let for subsets (cluster) $S \in \Omega$, $\bar{S} = C \setminus S$, and $\delta(S) = \{(i, j) \in E : i \in S \wedge j \notin S \vee i \notin S \wedge j \in S\}$. Significant $a(S) = \sum_{i \in S} a_i$ – total flow of customers in the cluster S.

Let's formulate the problem. It is necessary to find the minimum function

$$\sum_{(i,j) \in E} c_{ij} x_{ij} \tag{23}$$

subject to

$$\sum_{j \in N'} (y_{ji} - y_{ij}) = 2a_i, \ \forall i \in C, \tag{24}$$

$$\sum_{i \in V} \sum_{j \in C} y_{ij} = a(C), \ \sum_{j \in C} y_{jn'} = 0, \tag{25}$$

$$\sum_{\{i,j\} \in \delta(p)} x_{ij} = 2, \ \forall p \in C, \tag{26}$$

$$\sum_{i \in V} \sum_{j \in C} x_{ij} = \sum_{j \in C} x_{jn'}, \tag{27}$$

$$y_{ij} + y_{ji} = Q_K x_{ij}, \ \forall (i, j) \in E, \tag{28}$$

$$\sum_{\{i,j\} \in \delta(S)} x_{ij} \ge \sum_{i \in V} \sum_{j \in C} x_{ij}, \ S \subseteq V \cup C, \ V \subset S, \tag{29}$$

$$y_{ij} \le Q_{\pi(i)}, \ \forall i \in V, \ \forall j \in C, \tag{30}$$

$$y_{ij}, y_{ji} \ge 0 \text{ and integer, } x_{ij} \in \{0,1\} \ \forall (i, j) \in E. \tag{31}$$

Constraints (24), (25), and (31) determine the allowable flow. The constraints (26) ensure that in any valid solution, each client will have only two incident arcs. Conditions (27) mean that if $q = \sum_{i \in V} \sum_{j \in C} x_{ij}$ vehicles have left a set of nodes V, All of them must come to the common depot. The constraints (28) establish a relationship between the variables in the admissible solution of the problem, and the conditions (29) prohibit the formation of simple paths that begin and end in the nodes of the set V. The load capacity limit for different types of vehicles is set by conditions (30).

In [21] the theorem is proved that the set of admissible solutions to the problem (24)–(31) uniquely corresponds to the set of solutions of the FSMF problem (a problem with independent prices of arcs depending on the type of vehicle).

6. Formulation of the problem of constructing delivery routes for the transportation of cargo from the main hub to customers. Models on a set partition

Consider the HVRP problem statements proposed in [22] and [7] and based on the early work of M. Balinski and R. Quandt [23]. Let's define the route of the vehicle as a pair (R,k), де $R = (i_1, i_2, ..., i_{|R|})$, $i_1 = i_{|R|} = 0$, $\{i_2, i_3, ..., i_{|R|-1}\} \subseteq C$, $\left|\{i_2, i_3, ..., i_{|R|-1}\}\right| \geq 1$ represents a simple cycle that begins and ends in the depot, and k – determines the type of vehicle associated with R. Link to the route R is used to refer to both the sequence of customers visited and a subset of those customers, including the depot. Route (R,k) is permissible if the total request of customers visited on this route is absorbed into the carrying capacity of the vehicle Q_k, allocated to this route, $\sum_{h=2}^{|R|-1} a_{i_h} \leq Q_k$. Costs on the route (R,k) Include F_k (total costs for the purchase and maintenance of a vehicle of the type k) and the amount of costs on the arcs of the route $\sum_{h=1}^{|R|-1} f^k_{i_h i_{h+1}}$, $\forall k \in V$. Let R_k represents the set of all possible valid routes for a vehicle of type $k \in V$. For each route $l \in R_k$ let's denote f_{lk} – associated route costs (the sum of costs on the route arcs), f_l^k – the cost of unloading cargo to customers on the route.

Let $R_{ik} \subset R_k$ – a subset of vehicle routes of type k, that cover customers $i \in C$, and r_{lk}, defines a subset of customers visited on a route l, $r_{lk} = \{i_1, i_2, ..., i_h\}$, $\{i_1, i_2, ..., i_h\} \subseteq C$. Define $a_{lk} = \sum_{i \in r_{lk}} a_i$, $\forall l \in R_k$, $\forall k \in V$. Let's introduce Boolean variables $x_{lk} = 1$, if and only if the route $l \in R_k$ included in the optimal solution and $x_{lk} = 0$ otherwise.

You need to find the minimum

$$\sum_{k \in V} \sum_{l \in R_k} (F_k + f_{lk}) x_{lk} + 2 \sum_{k \in V} \sum_{l \in R_k} f_l^k (a_{lk} x_{lk}) \tag{32}$$

subject to

$$\sum_{k \in V} \sum_{l \in R_{ik}} x_{lk} = 1, \ \forall i \in C, \tag{33}$$

$$\sum_{l \in R_k} x_{lk} \leq m_k, \ \forall k \in V, \tag{34}$$

$$x_{lk} \in \{0,1\}, \ \forall l \in R_k, \ \forall k \in V. \tag{35}$$

The second component of the objective function is multiplied by two, since all loads unloaded from vehicles of type k on the route l, assigned to customers i, are loaded into the same vehicles at the depot, and the costs of loading and unloading a unit of cargo are assumed to be the same for vehicles of the same type. Constraints (33) ensure that each customer can only be served by one vehicle, and conditions (34) limit the number of vehicles of each type used (these conditions may not be present with an unlimited operating fleet).

Here is a slightly modified wording [7], which takes into account the time of customer service and constraints on the total time of movement of the vehicle. Let's define the set $\tilde{V}=\{1,2,...,K\}$ all types of vehicles (the number of vehicles of each type is not limited), numbered with an index $k=\overline{1,K}$. Let F_k, Q_k and T_k, respectively mean the cost of acquisition and maintenance, the load capacity and the maximum driving time of the type of vehicle $k\in\tilde{V}$. Let's introduce a variable for solving the problem v, indicating the total number of vehicles included in the working fleet $V=\{1,2,...,v\}$, $V\subseteq\tilde{V}$. Let's introduce variable costs α_k and the time factor β_k per unit distance for each type of vehicle $k\in\tilde{V}$. Let t_i – customer service time $i\in C$, R_p – the set of customers served by a vehicle with a number $p\in V$, and σ – function $\sigma:V\to\tilde{V}$, uniquely displays a set of numbers with V for a variety of vehicle types $\tilde{V}$ and $\sigma(p)$ indicates the type of vehicle with the lowest payload capacity $Q_{\sigma(p)}$, which can serve customers with R_p. Let π_p is the shortest traveling salesman's path (TSP route) with a starting and ending point at a depot that serves customers from R_p, and $\pi_p(i)$, indicates the position of the client i in π_p. Let's enter for the route π_p denomination: $D(\pi_p)$ – total distance, $T(\pi_p)$ – total travel time, $C(\pi_p)$ – the sum of variable and fixed cost. Let's denote by l_{ij} distance between customers i and j. Let S is a valid solution and is defined as $S=\{R_1,...,R_v\}$, and Π represents the set of all traveling salesman routes in S, $\Pi=\{\pi_1,...,\pi_v\}$.

Let's formulate the following optimization problem:

$$\underset{S,\Pi,v}{C(S)}=\sum_{p\in V}C(\pi_p) \tag{36}$$

subject to

$$\bigcup_{p\in V}R_p=C,\ R_p\cap R_q=\varnothing,\ \forall p\neq q\in V, \tag{37}$$

$$\sum_{i\in R_p}a_i\le Q_{\sigma(p)},\ \forall p\in V, \tag{38}$$

$$D(\pi_p)=\sum_{i\in R_p\cup\{0\}}l_{i\pi_p(i)},\ \forall p\in V, \tag{39}$$

$$T(\pi_p)=\beta_{\sigma(p)}D(\pi_p)+\sum_{i\in R_p}t_i\le T_{\sigma(p)},\ \forall p\in V, \tag{40}$$

$$C(\pi_p)=F_{\sigma(p)}+f_0^{\sigma(p)}(\sum_{i\in R_p}a_i)+\sum_{i\in R_p}f_i^{\sigma(p)}(a_i)+\alpha_{\sigma(p)}D(\pi_p),\ \forall p\in V. \tag{41}$$

Instead of a term $\alpha_{\sigma(p)}D(\pi_p)$ can be used $\sum_{i\in R_p\cup\{0\}}f^{\sigma(p)}_{i\pi_p(i)i\pi_p(i)}$, and instead of $\beta_{\sigma(p)}D(\pi_p)$ – $D(\pi_p)/\theta_{\sigma(p)}$, where $\theta_{\sigma(p)}$ – average speed of a vehicle type $\sigma(p)$. The objective function (36) determines the total cost on all routes. Constraints (37) and (38) mean that each customer is served by only one vehicle route and that the total volume of customer claims served by each route may not exceed the payload capacity of the dedicated vehicle. Equations (39) and (41) determine the total length and costs for each route, respectively. Constraints (40) mean that travel times for each route must not exceed a specified maximum value.

As noted, there are currently three exact algorithms for solving problems with a heterogeneous fleet of vehicles. Choi and Tcha (2007) [24] developed an algorithm based on the column generation method for linear relaxation of the original problem. They modified several dynamic programming algorithms to solve the classic VRP problem in order to efficiently generate valid columns, and then applied the branch and bounds procedure to arrive at an integer solution. The results of numerical experimentation with the developed algorithms showed their superiority in comparison with existing algorithms in terms of the quality of the solutions obtained and the time of calculations. Baldacci and Mingozzi [15] (2009) proposed a universal precision algorithm that uses the set partitioning method to solve all problems in the FCRPF family. They used three types of procedures based on LP and Lagrangian relaxation of the original problem and obtained new, more accurate lower bounds for solving problems. The third exact algorithm was proposed by Baldacci, Bartolini, Mingozzi and Roberti [16] (2010). It combines several iterative dual procedures to generate near-optimal dual non-integer solutions for a set of partitions (cluster solutions) and adds efficient cut-off inequalities to the coordinating problem to reduce the brute force in the column generation algorithm and obtain an integer solution.

For a final dual solution, a valid domain containing only integer solutions is defined. As stated in [16], this algorithm is more efficient than other known precision algorithms.

7. Mathematical model of the problem of constructing delivery routes for the transportation of cargo from the customers to depot

Let's formulate the problem in the same way (1)–(8). Let for each client j, $j=\overline{1,n}$, set average daily quantity $b_j > 0$, $b_j \in Z^+$ units of discrete cargo of a uniform size, which must be sent from the client to the main hub (depot) within a day. Let's introduce continuous variables z_{ji}, that determine the amount of cargo in the vehicle when it passes to the client i after the customer's visit j, $i, j \in N$ and Boolean variables x_{ij}^k, $x_{ij}^k = 1$, if the vehicle is of the type k moves away from the customer i to the client j and $x_{ij}^k = 0$ otherwise. Let $\sum_{j=1}^{n} x_{0j}^k$ is the total number of vehicles of the type used k.

It is necessary to find the minimum function

$$F_{CR} = \sum_{k\in V} F_k \sum_{j\in C} x_{0j}^k + \sum_{k\in V}\sum_{j\in N} f_j^k \Big(\sum_{i\in C} \big|(z_{ji} - z_{ij})\big| x_{ij}^k\Big) + \sum_{k\in V}\sum_{i,j\in N} f_{ij}^k x_{ij}^k , \tag{42}$$

subject to

$$\sum_{k\in V}\sum_{i\in N} x_{ij}^k = 1, \ \forall j \in C, \tag{43}$$

$$\sum_{i\in N} x_{ij}^k - \sum_{i\in N} x_{ji}^k = 0, \ \forall j \in C, \ \forall k \in V, \tag{44}$$

$$\sum_{j\in C} z_{0j} = 0, \ \sum_{j\in C} z_{j0} = \sum_{j\in C} b_j , \tag{45}$$

$$\sum_{i\in N} z_{ji} - \sum_{i\in N} z_{ij} = b_j , \ \forall j \in C, \tag{46}$$

$$z_{j0} \le \sum_{k\in V} Q_k x_{j0}^k , \ \forall j \in C, \tag{47}$$

$$z_{ij} \le \sum_{k\in V} (Q_k - b_j) x_{ij}^k , \ \forall i \in C, \ \forall j \in N, \ i \ne j, \tag{48}$$

$$x_{ij}^k \in \{0,1\}, \ z_{ij} \ge 0 \text{ and integer}, \ \forall i, j \in N, \ \forall k \in V. \tag{49}$$

Constraints (48) can be written as

$$b_i x_{ij}^k \le z_{ij} \le (Q_k - b_j) x_{ij}^k, \ \forall i \in C, \ \forall j \in N, \ i \ne j, \ \forall k \in V.$$

As in problem (1)–(8), in the statement of problem (42)–(49) constraints can be introduced on the number of vehicles of each type and on the maximum duration of routes in time. The first part of the objective function (42) defines the fixed costs for the purchase and maintenance of the vehicles used, the second is the cost of loading goods at customers and unloading them into the depot, and the third is the variable costs of delivering all cargo to the depot. Constraints (43) and (44) ensure that each customer j visited only once by any vehicle of the type k, and this vehicle, upon arrival at the client and loading of his cargo b_j must leave the client. Constraints (45) mean that the total amount of cargo at the exit from the depot is zero, and at the entrance to the depot is equal to the total requirements of all customers. Constraints (46) mean that with a customer visit j cargo b_j must be loaded, i.e. the amount of cargo in the vehicle after visiting the customer j increases by an amount b_j. These constraints ensure that the requirements of all customers are met and eliminate cycles that do not go through the depot. Constraints (47) ensure that the load capacity of vehicles is not exceeded. Constraints (48) bind variables z_{ij} and x_{ij}^k, and mean that no cargoes are transported from the i to j, if no vehicle maintains communication between these nodes, i.e. if $x_{ij}^k = 0$ $\forall k \in V$. Constraints (49) set the scope of variables.

It should be noted that mathematical models similar to those given in subsections 4 to 6 can also be formulated for collection routes.

Conclusion

1. For effective management of the processes of handling and transportation of discrete cargoes in the internal zones of the central trunk nodes of the global hierarchical network, public and private transport enterprises should optimize long-term, tactical and operational decisions using modern methods of operations research, combinatorial optimization and information-analytical decision support systems (DSS). Reduction of operating costs due to optimization of solutions allows to reduce tariffs for the transportation of cargo, maintain healthy competition among carriers for the provision of transport services and constantly improve the quality of service for economic enterprises and the population.

2. When determining the structure of the intra-junction transportation network and its mathematical model, the real geographical features and characteristics of the road sections of the transport network, external factors that are difficult to formalize and possible consequences on transport processes from unforeseen natural disasters should be taken into account. The model of the physical structure of the network should be formed with the participation of experienced experts and transport dispatchers for each central node. For large central hubs, the task should be set for the territorial location of several depots and the binding of the customer nodes served to them.

3. In most of the well-known works devoted to solving the problems of routing in transport, idealized mathematical models are considered, which do not take into account many of the limitations inherent in the real processes of cargo handling and transportation. The Euclidean distance between the nodes of the transportation network is often accepted when the triangle rule is fulfilled, and such important parameters as distance, travel time, transportation costs, service time, and others are present in some abstract form and are modeled by constants. The formulation of the tasks of intra-junction transportation should contain all the constraints and parameters that will allow to calculate close to the actual technical and economic indicators of the functioning of the transportation network in the internal zones of the central nodes.

4. The basic principles and schemes of organization of transportation in the inner zones of the central nodes are proposed, and the technical and economic features of real transport processes of handling and

transportation of discrete cargoes are determined, which should be taken into account in the formation of target functions of mathematical models of routing problems at the levels of long-term and current planning and operational management.

5. Classical heuristic and metaheuristic algorithms are most often used to solve the problems of constructing routes with a heterogeneous fleet of vehicles, which is explained on the one hand by the complexity of the problems under consideration, and on the other hand by the relatively low labor intensity of the development of such algorithms. However, it should be borne in mind that most heuristic algorithms on different instances of individual optimization problems can give solutions that are arbitrarily different from the global optimum. Therefore, to solve problems of large dimensions (more than 100 clients), it is better to use hybrid algorithms, which combine in various combinations exact (branches and boundaries, branches and clippings, branches of clipping and prices, column generation, set splitting, dynamic programming) and numerous heuristic and metaheuristic methods and approaches. Today, it can be argued that the development of hybrid algorithms for solving NP – complex cluster routing problems has become generally recognized in the world practice [25, 26]. In recent years, there has also been a tendency to build unified algorithms and portal-servers capable of solving a large class of routing problems with the ability to take into account many real constraints and parameters [27].

6. On the basis of the review and analysis of known mathematical models, several new variants of mathematical formulation of problems of designing routes of vehicles for the transportation of discrete cargo in the internal zones of the central nodes of the network have been developed. To solve the problems, precise, heuristic and metaheuristic methods and algorithms can be used, implemented in many commercial and non-commercial packages of mixed and integer programming programs, for example, IBM ILOG CPLEX, GAMS, AIMMS, Gurobi Optimizer [28], ABACUS, COIN-OR, GLPK, lp_solve [29]. Many of them are available for free on the NEOS server (https://neos-server.org/neos/).

Taking into account the hierarchical structure of the global (backbone) transportation network and, as a result, the small number of customer nodes in the internal zones of the central trunk nodes, preference should be given to precise and hybrid methods and algorithms. In particular, the algorithms proposed by R. Baldacci and others can be successfully used [14–17], which are currently considered to be one of the best accurate universal algorithms capable of finding optimal solutions to many routing problems with up to 100 clients.

7. Promising directions for solving routing problems include the development of stochastic models and algorithms for long-term planning, taking into account the risks of investing in the development of the vehicle fleet, dynamic models of current planning and operational management to determine the constraints of economic efficiency of the solutions already obtained for given intervals in case of fluctuations in cargo flows and changes in the parameters of the transport model. In this case, operational information can be received by the DSS from GPS from vehicles, electronic ordering systems via the Internet and mobile communication devices (cell phones, tablets, smartphones, etc.). Of great interest is also the creation of a single nationwide database in a standardized format based on one structured language (for example, XML) with complex test cases for different classes of typical routing problems.

8. The materials of the article form the methodological basis for the development of modern mathematical support for solving the problems of long-term, current and operational planning and management in the internal zones of the central trunk nodes of the global hierarchical network [30–33].

Authorship contribution statement

Volodymyr Vasyanin: Research and development of the article, Formulated the research problem, Writing the manuscript, Proposing future research directions.

Liudmyla Ushakova: Research and development of the article, Conducted analyses of get results, Writing the manuscript.

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Mathematical Models of the Problem of Constructing Delivery Routes of Cargo in the Internal Zones of Trunk Nodes of a Hierarchical Transport Network

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Introduction. The article discusses mathematical models of problems of constructing circular routes of vehicles in a multicommodity hierarchical network. As a rule, such networks consist of a decentralized backbone network and networks in the internal service areas of the backbone nodes (internal networks). In multicommodity networks, each node can exchange products (goods, cargo) with other nodes. In contrast to the distribution problems of a homogeneous interchangeable product, in multicommodity problems the flows of products are not interchangeable, the flow of each product must be delivered from a specific primary object to a specific customer. It is assumed that the multi-level structure of the transport network is defined and the geographical location of the main hubs and its internal service areas with a set of nodes for the delivery and collection of goods (customers) are known. Therefore, the problems of determining the main routes of vehicles and constructing circular routes of internal vehicles are considered independently of each other. The types of costs of real transport processes, which should be taken into account in the formation of the objective function of routing problems, are discussed and mathematical models of problems for constructing circular delivery routes with a heterogeneous fleet of vehicles are proposed. The possibility of solving the formulated problems with the help of well-known packages of mixed and integer linear programming is noted.

Purpose. The aim of the article is to formulate new mathematical models of the problem of constructing circular routes of vehicles in the internal networks of servicing the main nodes, which take into account the real costs of transport processes and the geographical features of internal networks.

The technique. The research methodology is based on the system analysis of many modern models, methods and algorithms for solving the problems of constructing circular routes for customer service in the internal zones of the main nodes of the hierarchical network.

Results. On the basis of the review and analysis of known mathematical models, several new variants of mathematical formulation of problems of designing routes of vehicles for the transportation of discrete cargo in the internal zones of the central nodes of the network have been developed. To solve the problems, precise, heuristic and metaheuristic methods and algorithms can be used, implemented in many commercial and non-commercial packages of mixed and integer programming programs, for example, IBM ILOG CPLEX, GAMS, AIMMS, Gurobi Optimizer, ABACUS, COIN-OR, GLPK, lp_solve. Many of them are available for free on the NEOS server (https://neos-server.org/neos/).

Scientific novelty and practical significance. The novelty of the work lies in the formulation of mathematical models of the problem of constructing circular routes of vehicles, which take into account the real costs of transport processes and geographical features of internal networks. The materials of the article form the methodological basis for the development of modern mathematical support for solving the problems of long-term, current and operational planning and management in the internal zones of the trunk nodes of the global hierarchical network.

Keywords: problems of combinatorial optimization, mathematical models of circular routes of vehicles.

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Математичні моделі задачі побудови маршрутів доставки вантажів у внутрішніх зонах магістральних вузлів ієрархічної транспортної мережі

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Вступ. У статті розглядаються математичні моделі задач побудови кільцевих маршрутів транспортних засобів у багатопродуктовій ієрархічній мережі. Як правило, такі мережі складаються з децентралізованої магістральної мережі та мереж у внутрішніх зонах обслуговування магістральних вузлів (внутрішні мережі). У багатопродуктових мережах кожен вузол може обмінюватися продуктами (товарами, вантажами) з іншими вузлами. На відміну від задач розподілу однорідного взаємозамінного продукту, у багатопродуктових задачах потоки продуктів не взаємозамінні, потік кожного продукту має бути доставлений з певного первинного об’єкта до конкретного клієнта. Передбачається, що багаторівнева структура транспортної мережі визначена і відомі географічне розташування магістральних вузлів та його внутрішні зони обслуговування з множиною вузлів доставки і збору вантажів (клієнтів). Тому задачі визначення магістральних маршрутів транспортних засобів та побудови кільцевих маршрутів внутрішніх транспортних засобів розглядаються незалежно одна від одної. Обговорюються види витрат реальних транспортних процесів, які мають враховуватися при формуванні цільової функції задач маршрутизації та запропоновані математичні моделі задач побудови кільцевих маршрутів доставки із неоднорідним парком транспортних засобів. Зазначається можливість розв’язання сформульованих задач за допомогою відомих пакетів змішаного та цілочислового лінійного програмування.

Мета роботи полягає у формулюванні нових математичних моделей задачі побудови кільцевих маршрутів транспортних засобів у внутрішніх мережах обслуговування магістральних вузлів, у яких враховуються реальні витрати транспортних процесів та географічні особливості внутрішніх мереж.

Методика досліджень заснована на системному аналізу багатьох сучасних моделей, методів та алгоритмів вирішення задач побудови кільцевих маршрутів для обслуговування клієнтів у внутрішніх зонах магістральних вузлів ієрархічній мережі.

Результати. На основі огляду та аналізу відомих математичних моделей розроблено декілька нових варіантів математичної постановки задач проектування маршрутів транспортних засобів для перевезення дискретних вантажів у внутрішніх зонах центральних вузлів мережі. Для вирішення поставлених завдань можуть бути використані точні, евристичні та метаевристичні методи і алгоритми, реалізовані в багатьох комерційних і некомерційних пакетах програм змішаного і цілочислового програмування, наприклад, IBM ILOG CPLEX, GAMS, AIMMS, Gurobi Optimizer, ABACUS, COIN-OR, GLPK, lp_solve. Багато з них доступні безкоштовно на сервері NEOS (https://neos-server.org/neos/).

Наукова новизна і практична значимість. Новизна роботи полягає у формулюванні математичних моделей задачі побудови кільцевих маршрутів транспортних засобів, у яких враховуються реальні витрати транспортних процесів та географічні особливості внутрішніх мереж. Матеріали статті формують методологічну основу для розробки сучасного математичного забезпечення розв'язання задач довгострокового, поточного та оперативного планування та управління у внутрішніх зонах магістральних вузлів глобальної ієрархічної мережі.

Ключові слова: задачі комбінаторної оптимізації, математичні моделі кільцевих маршрутів транспортних засобів.