CYBERNETICS
and COMPUTER
TECHNOLOGIES

UDC 519.6
DOI:10.34229/2707-451X.25.3.1

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SYNTHESIS OF POWER AND MOTION CONTROL OF PLATE HEATING SOURCES AND OPTIMIZATION OF TEMPERATURE MEASUREMENT POINTS PLACEMENT

This paper investigates the problem of synthesizing optimal control of moving point heat sources for heating a two-dimensional plate. The powers of the sources and their trajectories of motion, described by ordinary differential equations, are optimized. In addition, in the problem under consideration, the locations of temperature measurement points are also optimized. The necessary optimality conditions for the feedback parameters and the coordinates for setting the measurement points are obtained. The conditions contain formulas for the components of the gradient of the objective functional according to the parameters being optimized. The results of computer experiments obtained using first-order numerical optimization methods are presented.

Keywords: *plate heating, feedback control, moving sources, temperature measurement points, feedback parameters.*

Introduction. The problem of optimal synthesis of control actions for the motion and power of point heat sources during heating of a two-dimensional thin plate is considered. The current values of the control actions depend on the measured values of the plate temperature at the points where the measuring devices are installed. The main specific feature of the problem under consideration is that it also requires determining the optimal placements for the measurement devices.

The problem under consideration belongs to the problems of optimal control of lumped sources in systems with distributed parameters [1–6]. Such problems are described by initial-boundary–value problems with respect to partial derivative equations of various types. The theory of optimal control problems for systems with distributed parameters began to be most actively developed in the 60’s of the last century in the works [5, 7, 8]. This was due to the need to problem such important processes as the development of large oil and gas fields and pipeline transportation of hydrocarbon raw materials. Similar problems are relevant in metallurgy, ecology and many other industries.

A special place in the theory and practice of optimal control is occupied by control problems with feedback from the object [4, 7, 9, 10]. At the end of the XIX and beginning of the XX centuries, special devices were developed to control and regulate some industrial processes and technical objects based on the first theoretical studies of J.C. Maxwell, E.J. Routh, I.A. Vyshnegradsky, A. Gurvits, A.M. Lyapunov and other scientists and engineers. In the 1950s of the last century, the results of research by L.S. Pontryagin [11], R.E. Bellman [12], A.M. Letov [13] and many other scientists [14, 15] made it possible to solve serious problems in rocket science and cosmonautics (astronautics) and create automatic control systems for various technological processes and industrial facilities.

The works [4, 15, 16] provide a fairly in-depth analysis of the development and current state of control theory and its applications.

The novelty of the formulation and approach to solving the problem of synthesizing optimal control of lumped sources for a system with distributed parameters considered in this article is as follows. Firstly, for the first time, a simultaneous synthesis of control actions by the power of moving sources and the trajectory of their motion itself is carried out by current measurements. Secondly, the problem of optimal placement of process state measurement devices on the plate is posed and solved. Thirdly, for the synthesized control actions, a dependence is proposed that is linear in terms of the measured state values. Fourthly, the problem of synthesis of control actions is reduced to a problem of finite-dimensional conditional optimization with linear constraints.

The necessary conditions of optimality of the synthesized feedback parameters and formulas for the components of the gradient of the objective functional on these parameters are obtained. These formulas allow the use of efficient first-order numerical optimization methods to solve the problem of feedback parameter synthesis. The results of computer experiments obtained in solving an illustrative problem are presented.

The approach proposed in this paper can be used in constructing various automatic control and regulation systems for lumped sources for objects with distributed parameters described by other types of initial-boundary–value problems.

Problem statement. The problem of controlling the heating process of a thin plate described by a differential equation of parabolic type is considered [17]:

$$u_t(x,t) = a^2 \mathrm{div}(\mathrm{grad} u(x,t)) - \lambda_0 [u(x,t) - \theta] + \sum_{i=1}^{N_c} q_i(t) \delta(x - z_i(t)), \quad x \in \Omega, \quad t \in (t_0, t_f], \tag{1}$$

with initial and boundary conditions

$$u(x, t_0) = b(x) = b, \quad x \in \bar{\Omega} = \Omega \bigcup \Gamma, \tag{2}$$

$$\frac{\partial u(x,t)}{\partial \mathrm{n}} = \lambda [u(x,t) - \theta], \quad x \in \Gamma, \ \ t \in (t_0, t_f]. \tag{3}$$

Here $u(x,t)$ is plate temperature at point $x \in \bar{\Omega} \subset \mathrm{R}^2$ at time t; Ω is the domain occupied by the plate, with the boundary of Γ; n is the internal normal to Γ; $a^2 > 0$, $\lambda_0 \geq 0$, $\lambda \geq 0$ are specified parameters of the heating process; θ is ambient temperature; $\delta(\cdot)$ is a two-dimensional Dirac function for which, for arbitrary continuous functions in Ω, $f(x)$ and $\tilde{x} \in \Omega$ takes place:

$$\iint_{\Omega} f(x) \delta(x - \tilde{x}) dx = f(\tilde{x}), \quad \iint_{\Omega} \delta(x) dx = 1.$$

The heating of the plate is produced with controlled power by N_c point moving sources, determined by piecewise continuous functions $q(t) = (q_1(t), q_2(t), \ldots, q_{N_c}(t))$ satisfying the conditions:

$$Q = \{q_i(t): \ \ \underline{q_i} \leq q_i(t) \leq \overline{q_i}, \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c\}, \tag{4}$$

here $\underline{q_i}$, $\overline{q_i}$, $i = 1, 2, \ldots, N_c$ are given.

The motion of the sources is described by differential equations with ordinary derivatives:

$$\dot{z}_i(t) = f_i(z_i(t), \vartheta_i(t)), \quad t \in (t_0, t_f], \quad i = 1, 2, \ldots, N_c, \tag{5}$$

$$z_i(t_0) = \breve{z}_i, \quad i = 1,2,\ldots,N_c, \tag{6}$$

where $z_i(t) \in R^2$ are the coordinates of the location on the plate of the i-th source at the time t; $\breve{z}_i \in \Omega$, $i = 1,2,\ldots,N_c$ are given; $f_i(\cdot,\cdot)$, $i = 1,2,\ldots,N_c$ are given continuously differentiable two-dimensional vector functions. Piecewise continuous functions $\vartheta_i(t) \in R^r$, $r \le 2$, $i = 1,2,\ldots,N_c$, are control actions and determine the trajectories of the sources on the plate. There are constraints on the control $\vartheta_i(t) = (\vartheta_1(t),\ldots,\vartheta_r(t))$, $i = 1,2,\ldots,N_c$ of the motion of sources:

$$V = \{\vartheta_i^\gamma(t): \ \underline{\vartheta_i^\gamma} \le \vartheta_i^\gamma(t) \le \overline{\vartheta_i^\gamma}, \quad t \in [t_0,t_f], \quad \gamma = 1,\ldots,r, \quad i = 1,2,\ldots,N_c\}, \tag{7}$$

where $\underline{\vartheta_i^\gamma}$, $\overline{\vartheta_i^\gamma}$, $\gamma = 1,\ldots,r$, $i = 1,2,\ldots,N_c$, are given values.

The source trajectories must satisfy the natural constraint:

$$z_i(t) = \Omega, \quad t \in [t_0,t_f], \quad i = 1,2,\ldots,N_c. \tag{8}$$

The solution of the initial-boundary–value problem (1)–(3) will be understood in a generalized sense, i.e. as a function from the class $u(x,t) \in H^{1,1}\left(\Omega \times [t_0,t_f]\right)$. In the works [17, 18], under the accepted assumptions for the data of the problem (1)–(3), the existence and uniqueness of a generalized solution in the problem (1)–(3) is shown.

Condition (2) means that at the initial moment of time t_0 the temperature at all points of the plate is the same, but is not exactly specified, and it is known that it belongs to some given set $B \subset R$. We will assume that $\rho_B(b)$ is known the density distribution function for which the conditions are satisfied

$$\rho_B(b) \ge 0, \quad b \in B, \quad \int_B \rho_B(b)db = 1.$$

The value of the external ambient temperature θ, participating in the equation (1) and condition (3), does not change during the heating process and is defined on a given set Θ with a known density distribution function $\rho_\Theta(\theta)$ such that

$$\rho_\Theta(\theta) \ge 0, \quad \theta \in \Theta, \quad \int_\Theta \rho_\Theta(\theta)d\theta = 1.$$

The problem of optimal control of the plate heating process under consideration consists of finding the admissible values of the control actions on the motion $\vartheta = \vartheta(t) = (\vartheta_1(t),\vartheta_2(t),\ldots,\vartheta_{N_c}(t)) \in V$ and the powers of sources $q = q(t) = (q_1(t),q_2(t),\ldots,q_{N_c}(t)) \in Q$, delivering, on average over all possible values of the initial states $b \in B$ and the ambient temperature $\theta \in \Theta$, the minimum value of the following objective functional:

$$J(q,\vartheta) = \int_B \int_\Theta I(q,\vartheta;b,\theta)\rho_\Theta(\theta)\rho_B(b)d\theta db, \tag{9}$$

$$I(q,\vartheta;b,\theta) = \iint_\Omega \mu(x)\left[u(x,t_f) - U(x)\right]^2 dx + \varepsilon_1 \left\|q(t) - \hat{q}\right\|^2_{L_2^{N_c}[t_0,t_f]} + \varepsilon_2 \left\|\vartheta(t) - \hat{\vartheta}\right\|^2_{L_2^{N_c}[t_0,t_f]}. \tag{10}$$

Here $u(x,t) = u(x,t;q,\vartheta,b,\theta)$ is the solution of the initial-boundary-value problem (1)–(3) with the initial condition $u(x,t_0) = b$, the ambient temperature θ for admissible values of the source powers $q = q(t)$ and the effects on their movements $\vartheta = \vartheta(t)$; $U(x)$ is the given temperature distribution of the plate that must be achieved at the end of the heating process; $\mu(x) \geq 0$, $x \in \Omega$, is the given weight function; ε_1, ε_2, $\hat{q}$, $\hat{\vartheta}$ are the given regularization parameters of the functional of the problem [19].

The problem of controlling the plate heating process (1)–(10) under consideration can be interpreted as follows. Let $W(x,t;B,\Theta,q,\vartheta)$ be the set of solutions of the initial-boundary–value problem (1)–(3) for all possible values of the initial conditions $b \in B$ in (2) and the ambient temperature $\theta \in \Theta$ in (1), (3) for given admissible values of the power of the sources $q(t)$, and the actions $\vartheta(t)$ on their motion. Thus, the set $W(x,t;B,\Theta,q,\vartheta)$, $x \in \Omega$, $t \in [t_0, t_f]$ represents a «bundle» of trajectories of the phase state of the heating process for a given control pair $(q(t), \vartheta(t))$. Then the objective functional (9), (10) determines the quality of the choice of the control pair for controlling the «trajectory bundle» $W(x,t;B,\Theta,q,\vartheta)$.

Now let us reformulate the problem in the case of feedback.

Let sensors be installed at points of the plate $\xi_j = (\xi_{j,1}, \xi_{j,2}) \in \Omega$, $j = 1,2,\ldots,N_o$ to measure the current temperature values at these points of the plate continuously throughout the entire heating process:

$$\hat{u}_j = u(\xi_j, t), \quad t \in [t_0, t_f], \quad j = 1,2,\ldots,N_o. \tag{11}$$

The measurement results are proposed to be used to form the current values of control actions for each of the sources $(q_i(t), \vartheta_i(t))$, $i = 1,2,\ldots,N_c$. For this purpose, we will consider the following linear feedback of control actions from the measured temperature values at the measurement points:

$$q_i(t) = \sum_{j=1}^{N_o} \alpha^1_{i,j} \left[u(\xi_j, t) - \beta^1_{i,j} \right], \quad t \in [t_0, t_f], \quad i = 1,2,\ldots,N_c, \tag{12}$$

$$\vartheta^\gamma_i(t) = \sum_{j=1}^{N_o} \alpha^{2,\gamma}_{i,j} \left[u(\xi_j, t) - \beta^{2,\gamma}_{i,j} \right], \quad t \in [t_0, t_f], \quad \gamma = 1,\ldots,r, \quad i = 1,2,\ldots,N_c. \tag{13}$$

Here $\alpha^1_{i,j}$, $\beta^1_{i,j}$, $\alpha^{2,\gamma}_{i,j}$, $\beta^{2,\gamma}_{i,j}$ are feedback parameters, ξ_j are the locations of the measurement sensors, $\gamma = 1,\ldots,r$, $i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$.

In (12), (13) the quantities $\alpha^1_{i,j}$, $\alpha^{2,\gamma}_{i,j}$ determine the values of the gain factors for the i-th source relative to the deviation of the temperature value at the j-th measurement point from the nominal values $\beta^1_{i,j}$, $\beta^{2,\gamma}_{i,j}$ for the i-th source at the j-th measurement point. The optimized nominal values $\beta^1_{i,j}$, $\beta^{2,\gamma}_{i,j}$ are significant determined by the values of the given function $U(x)$ at the measurement points $x = \xi_j$, $j = 1,2,\ldots,N_o$.

From the properties of differential equations (1), (5) it follows that the control actions $(q(t), \vartheta(t))$, determined from (11), (12), (13) are themselves continuous functions of time.

Substituting the formulas of control actions with feedback (12), (13) into the differential equations (1) and (5), respectively, we obtain the following equations:

$$u_t(x,t) = a^2 \mathrm{div}(\mathrm{grad}\, u(x,t)) - \lambda_0 [u(x,t) - \theta] + \sum_{i=1}^{N_c} q_i(t) \delta(x - z_i(t)), \tag{14}$$

$$x \in \Omega, \quad t \in (t_0, t_f],$$

$$\dot{z}_i(t) = f_i\left(z_i(t), \sum_{j=1}^{N_o} \alpha_{i,j}^2 \left[u(\xi_j,t) - \beta_{i,j}^2 \right] \right), \quad t \in (t_0, t_f], \quad i = 1,2,\ldots,N_c. \tag{15}$$

The differential equation (14) is called point loaded due to the participation in it of the values of the sought function $u(x,t)$ at the measurement points at $x = \xi_j$, $j = 1,2,\ldots,N_o$ [20, 21]. Such initial-boundary–value problems are investigated in the works [22–24], in which conditions for the existence and uniqueness of their solution are obtained, and numerical methods for their solution are proposed and investigated.

We will assume that the locations of the measurement points ξ_j, $j = 1,2,\ldots,N_o$ are not specified and they need to be placed in an optimal way taking into account the optimality criterion specified by the functional (7), (8). Let there be constraints on the places of their installation from the technological and technical conditions:

$$\xi_j \in \Omega_j \subset \Omega, \quad j = 1,2,\ldots,N_o, \tag{16}$$

where closed subsets $\Omega_j \subset \Omega$, $j = 1,2,\ldots,N_o$ are given.

Thus, the problem requires determining the feedback parameters $\alpha^1 = ((\alpha_{i,j}^1))$, $\beta^1 = ((\beta_{i,j}^1))$, $\alpha^2 = ((\alpha_{i,j}^{2,1}),\ldots,(\alpha_{i,j}^{2,r}))$, $\beta^2 = ((\beta_{i,j}^{2,1}),\ldots,(\beta_{i,j}^{2,r}))$, and the locations of the measurement points ξ_j, $i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$. In this case, the following constraints must be met: on the power and velocity of motion of sources (4), (7), on the placement of measurement points (16), and the objective functionality must take the minimum possible value. The vector of parameters optimized in the problem is denoted by $y = (\alpha^1,\beta^1,\alpha^2,\beta^2,\xi) \in R^N$, $N = 2N_oN_c + 2rN_oN_c + 2N_o$, which consists of: $2N_oN_c$ parameters α^1,β^1, $2rN_oN_c$ parameters α^2,β^2, $2N_o$ parameters ξ. Taking into account these notations, we write the objective functional (9), (10) of the problem under consideration as follows:

$$J(y) = \int_B \int_\Theta I(y;b,\theta)\rho_\Theta(\theta)\rho_B(b)\,d\theta\,db, \tag{17}$$

$$I(y;b,\theta) = \iint_\Omega \mu(x)\left[u(x,t_f) - U(x)\right]^2 dx + \varepsilon_1 \|y - \hat{y}\|_{R^N}^2. \tag{18}$$

Here $u(x,t) = u(x,t;y,b,\theta)$ and $z(t) = z(t;y)$ are solutions, respectively, of the differential equation (14) and (15) for given feedback parameters $y = (\alpha^1,\beta^1,\alpha^2,\beta^2,\xi)$, initial condition $u(x,t_0) = b$, and ambient temperature θ; ε, $\hat{y}$ are given parameters of regularization of the functional of the problem [19].

The resulting problem (14), (2), (3), (4), (15), (6), (7), (8), (17), (18) refers to the problems of parametric optimal control of a distributed system with feedback. The searched vector in the problem is a finite-dimensional vector $y \in R^N$. Its specific features include the loading of the differential equation, the participation of the Dirac function in it, and also the fact that the value of the objective functional is determined not by a single solution of the initial-boundary–value problem, but by a bundle of solutions, provided that the initial condition and the ambient temperature take not a single value, but a set of values, respectively, from B and Θ.

Formulas for the gradient of the functional of the problem. Necessary conditions for optimality. For the numerical solution of the obtained parametric optimal control problem, it is proposed to use iterative methods of first-order optimization.

It is easy to verify that the functional of the original optimal control problem (1)–(10) is convex in $q(t)$, $\vartheta(t)$ for fixed $\xi \in R^{2N_o}$ places of measuring the process state. The functional of the problem (17), (18) with feedback (12), (13), as can be seen from (14), (15), in the general case is not convex with respect to the optimized feedback parameters y. This follows from the nonlinearity of the dependence of the solution of the boundary-value problem $u(x,t;y,b,\theta)$ on the parameters $y=(\alpha^1,\beta^1,\alpha^2,\beta^2,\xi)$. Therefore, the optimality conditions formulated in this section are only of a local character. The formulas for the components of the functional gradient involved in these conditions can be used in the numerical solution of the problem of determining the local optimal values of the feedback parameters or for local refinement of these values. To determine the optimal values of the feedback parameters, global optimization methods can be used together with methods of local improvement of the parameter values by first-order optimization methods using the functional gradient formulas obtained below.

We will write the constraints (4), (7) on the control actions using (12), (13) in the following form

$$\underline{q_i} \le \sum_{j=1}^{N_o} \alpha_{i,j}^1 \left[u(\xi_j,t) - \beta_{i,j}^1 \right] \le \overline{q_i}, \quad t \in [t_0,t_f], \quad i = 1,2,\ldots,N_c, \tag{19}$$

$$\underline{\vartheta_i^\gamma} \le \sum_{j=1}^{N_o} \alpha_{i,j}^{2,\gamma} \left[u(\xi_j,t) - \beta_{i,j}^{2,\gamma} \right] \le \overline{\vartheta_i^\gamma}, \quad t \in [t_0,t_f], \quad \gamma = 1,\ldots,r, \quad i = 1,2,\ldots,N_c. \tag{20}$$

Let us introduce the notation

$$g_i(t;y) = \left|\tilde{g}_i(t;y)\right| - \frac{\overline{q_i} - \underline{q_i}}{2}, \quad t \in [t_0,t_f], \quad i = 1,2,\ldots,N_c,$$

$$\tilde{g}_i(t;y) = \sum_{j=1}^{N_o} \alpha_{i,j}^1 \left[u(\xi_j,t) - \beta_{i,j}^1 \right] - \frac{\overline{q_i} + \underline{q_i}}{2}$$

$$w_i^\gamma(t;y) = \left|\tilde{w}_i^\gamma(t;y)\right| - \frac{\overline{\vartheta_i^\gamma} - \underline{\vartheta_i^\gamma}}{2}, \quad t \in [t_0,t_f], \quad \gamma = 1,\ldots,r, \quad i = 1,2,\ldots,N_c,$$

$$\tilde{w}_i^\gamma(t;y) = \sum_{j=1}^{N_o} \alpha_{i,j}^{2,\gamma} \left[u(\xi_j,t) - \beta_{i,j}^{2,\gamma} \right] - \frac{\overline{\vartheta_i^\gamma} + \underline{\vartheta_i^\gamma}}{2}.$$

Then the constraints (19) and (20) can be written as follows:

$$g_i(t;y) \le 0, \quad t \in [t_0,t_f], \quad i = 1,2,\ldots,N_c, \tag{21}$$

$$w_i^\gamma(t;y) \le 0, \quad t \in [t_0,t_f], \quad \gamma = 1,\ldots,r, \quad i = 1,2,\ldots,N_c. \tag{22}$$

To take into account the constraints (21) and (22) in the problem of synthesizing the parameters y, we use the external penalty functional method, adding to the functional (17), (18) a penalty term for violating the constraints (21) and (22):

$$J_{\Re}(y)=\int\limits_{B}\int\limits_{\Theta} I_{\Re}(y;b,\theta)\rho_{\Theta}(\theta)\rho_B(b)d\theta db\,, \tag{23}$$

$$I_{\Re}(y;b,\theta)=\iint\limits_{\Omega}\mu(x)\left[u(x,t_f)-U(x)\right]^2dx+\varepsilon_1\left\|y-\hat{y}\right\|^2_{R^N}+ \tag{24}$$

$$+\Re_1\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left[g_i^{+}(t;y)\right]^2dt+\Re_2\sum_{i=1}^{N_c}\sum_{\gamma=1}^{r}\int\limits_{t_0}^{t_f}\left[w_i^{\gamma,+}(t;y)\right]^2dt\,.$$

The functional (23) is minimized iteratively provided that the penalty coefficients $\Re=(\Re_1,\Re_2)$ tend to $+\infty$. The notations $g_i^{+}(t;y)$, $w_i^{\gamma,+}(t;y)$ mean that $g_i^{+}(t;y)=g_i(t;y)$, $w_i^{\gamma,+}(t;y)=w_i^{\gamma}(t;y)$ if $g_i(t;y)>0$, $w_i^{\gamma}(t;y)>0$ and $g_i^{+}(t;y)=0$, $w_i^{\gamma,+}(t;y)=0$ if $g_i(t;y)\le 0$, $w_i^{\gamma}(t;y)\le 0$, $\gamma=1,\ldots,r$, $i=1,2,\ldots,N_c$.

Next, we will use the function $\mathrm{sgn}(\cdot)$, which is equal to –1 for negative argument values, zero for an argument equal to 0, and 1 for positive argument values.

When minimizing the penalty functional (23), taking into account the positional constraints (16), given their simplicity, will be carried out using the projection operator onto the set defined by these constraints.

In general, to solve the problem of optimal synthesis of feedback parameters y, it is proposed to use iterative minimization methods with a combination of penalty function methods.

In particular, a gradient method with respect to the penalty function can be used, which we write in the form

$$y^{k+1}=\mathcal{P}_{(16)}\left[y^k-\nu^k\mathrm{grad}_y J_{\Re}(y^k)\right], \tag{25}$$

$$\nu^{k+1}=\arg\min_{\nu\ge 0} J_{\Re}\left(\mathcal{P}_{(16)}\left[y^k-\nu^k\mathrm{grad}_y J_{\Re}(y^k)\right]\right),\quad k=0,1,\ldots. \tag{26}$$

Here $\mathrm{grad}_y J_{\Re}(y^k)$ is the gradient of the penalty functional (23), (24), ν^k is the step size in the direction of the antigradient of the functional $J_{\Re}(y)$ [19].

It is clear that the main element required to implement the procedure (25) is the gradient of the functional $\mathrm{grad}_y J_{\Re}(y)$. The components of the gradient of the penalty functional with respect to the parameters of continuous feedback are determined from the following theorem.

Theorem 1. *The objective functional* $J_{\Re}(y)$ *of the problem (14), (2), (3), (15), (6), (17), (18) is differentiated with respect to the synthesized parameters* $y=(\alpha^1,\beta^1,\alpha^2,\beta^2,\xi)$ *with continuous feedback (11). The components of the gradient of the functional are determined by the formulas*

$$\frac{\partial J_{\Re}(y)}{\partial\alpha^1_{i,j}}=\int\limits_{B}\int\limits_{\Theta}\left\{-\int\limits_{t_0}^{t_f}\left(\psi(z_i(t),t)-2\Re_1\,\mathrm{sgn}(\tilde{g}_i(t;y))g_i^{+}(t;y)\right)\left[u(\xi_j,t)-\beta^1_{i,j}\right]dt+\right. \tag{27}$$

$$\left.+2\varepsilon\left(\alpha^1_{i,j}-\hat{\alpha}^1_{i,j}\right)\right\}\rho_{\Theta}(\theta)\rho_B(b)d\theta db\,,$$

$$\frac{\partial J_{\Re}(y)}{\partial \beta_{i,j}^{1}} = \int\limits_{B}\int\limits_{\Theta}\left\{-\int\limits_{t_0}^{t_f}\Big(\psi(z_i(t),t) - 2\Re_1\,\mathrm{sgn}(\tilde{g}_i(t;y))g_i^{+}(t;y)\Big)\alpha_{i,j}^{1}dt + \right. \tag{28}$$

$$\left. +2\varepsilon\Big(\beta_{i,j}^{1} - \hat{\beta}_{i,j}^{1}\Big)\right\}\rho_{\Theta}(\theta)\rho_B(b)d\theta db\,,$$

$$\frac{\partial J_{\Re}(y)}{\partial \alpha_{i,j}^{2,\gamma}} = \int\limits_{B}\int\limits_{\Theta}\left\{-\int\limits_{t_0}^{t_f}\left(\phi_i^{\mathrm{T}}(t)\frac{\partial f_i(z_i(t;y),\vartheta_i(t;y))}{\partial \vartheta_i^{\gamma}} - 2\Re_2\,\mathrm{sgn}(\tilde{w}_i^{\gamma}(t;y))w_i^{\gamma,+}(t;y)\right)\times\right. \tag{29}$$

$$\left.\times\Big[u(\xi_j,t) - \beta_{i,j}^{2,\gamma}\Big]dt + 2\varepsilon\Big(\alpha_{i,j}^{2,\gamma} - \hat{\alpha}_{i,j}^{2,\gamma}\Big)\right\}\rho_{\Theta}(\theta)\rho_B(b)d\theta db\,,$$

$$\frac{\partial J_{\Re}(y)}{\partial \beta_{i,j}^{2,\gamma}} = \int\limits_{B}\int\limits_{\Theta}\left\{-\int\limits_{t_0}^{t_f}\left(\phi_i^{\mathrm{T}}(t)\frac{\partial f_i(z_i(t;y),\vartheta_i(t;y))}{\partial \vartheta_i^{\gamma}} - 2\Re_2\,\mathrm{sgn}(\tilde{w}_i^{\gamma}(t;y))w_i^{\gamma,+}(t;y)\right)\alpha_{i,j}^{2,\gamma}dt + \right. \tag{30}$$

$$\left. +2\varepsilon\Big(\beta_{i,j}^{2,\gamma} - \hat{\beta}_{i,j}^{2,\gamma}\Big)\right\}\rho_{\Theta}(\theta)\rho_B(b)d\theta db\,,$$

$$\frac{\partial J_{\Re}(y)}{\partial \xi_{j,k}} = \int\limits_{B}\int\limits_{\Theta}\left\{-\sum_{i=1}^{N_c}\int\limits_{t_0}^{t_f}\left[\Big(\psi(z_i(t),t) - 2\Re_1\,\mathrm{sgn}(\tilde{g}_i(t;y))g_i^{+}(t;y)\Big)\alpha_{i,j}^{1} + \right.\right. \tag{31}$$

$$\left. + \sum_{\gamma=1}^{r}\left(\phi_i^{\mathrm{T}}(t)\frac{\partial f_i(z_i(t;y),\vartheta_i(t;y))}{\partial \vartheta_i^{\gamma}} - 2\Re_2\,\mathrm{sgn}(\tilde{w}_i^{\gamma}(t;y))w_i^{\gamma,+}(t;y)\right)\alpha_{i,j}^{2,\gamma}\right]u_{x_k}(\xi_j,t)dt +$$

$$\left. +2\varepsilon\Big(\xi_{j,k} - \hat{\xi}_{j,k}\Big)\right\}\rho_{\Theta}(\theta)\rho_B(b)d\theta db\,,$$

where $i = 1,2,\ldots,N_c$, $j = 1,2,\ldots,N_o$, $k = 1,2$, $\psi(x,t) = \psi(x,t;y,b,\theta)$ *and* $\phi_i(t) = \phi_i(t;y,b,\theta)$ *for the current parameter vector* y, *admissible initial condition* $b \in B$ *and ambient temperature* $\theta \in \Theta$ *are solutions of the following adjoint boundary value problem and the Cauchy problem, respectively:*

$$\psi_t(x,t) = -a^2\mathrm{div}(\mathrm{grad}\psi(x,t)) + \lambda_0\psi(x,t) \tag{32}$$

$$-\sum_{j=1}^{N_o}\delta(x-\xi_j)\sum_{i=1}^{N_c}\left[\Big(\psi(z_i(t),t) - 2\Re_1\,\mathrm{sgn}(\tilde{g}_i(t;y))g_i^{+}(t;y)\Big)\alpha_{i,j}^{1}\right.$$

$$\left. + \sum_{\gamma=1}^{r}\left(\phi_i^{\mathrm{T}}(t)\frac{\partial f_i(z_i(t;y),\vartheta_i(t;y))}{\partial \vartheta_i^{\gamma}} - 2\Re_2\,\mathrm{sgn}(\tilde{w}_i^{\gamma}(t;y))w_i^{\gamma,+}(t;y)\right)\alpha_{i,j}^{2,\gamma}\right], \quad x \in \Omega, \quad t \in [t_0, t_f),$$

$$\psi(x,t_f) = -2\mu(x)\Big[u(x,t_f) - U(x)\Big], \quad x \in \bar{\Omega}, \tag{33}$$

$$\frac{\partial \psi(x,t)}{\partial n} = \lambda\psi(x,t), \quad x \in \Gamma, \quad t \in [t_0, t_f), \tag{34}$$

$$\dot{\phi}_i(t) = -\left(\frac{\partial f_i(z_i(t;y), \vartheta_i(t;y))}{\partial z_i}\right)^{\mathrm{T}} \phi_i(t) - \sum_{i=1}^{N_c} \alpha_{i,j}^1 \left[u(\xi_j, t) - \beta_{i,j}^1\right] \mathrm{grad}_x \psi(z_i(t), t), \quad (35)$$

$$t \in [t_0, t_f), \quad i = 1, 2, \ldots, N_c,$$

$$\phi_i(t_f) = 0, \quad i = 1, 2, \ldots, N_c. \quad (36)$$

For further proof, a well-known technology for calculating the increment of the functional when receiving increments of the optimized feedback parameters is used [19].

Now let us formulate the necessary conditions for the optimality of the feedback parameters for the problems under consideration y^* [25].

Theorem 2. *For the optimality of the feedback parameters y^* in problem (14), (2), (3), (15), (6), (17), (18), (4), (7), (16), (21), (22) it is necessary and sufficient that the following condition be satisfied:*

$$\left\langle \mathrm{grad}_y J_{\Re}(y^*), y - y^* \right\rangle \le 0,$$

for all admissible values of the feedback parameters y satisfying the conditions (16), (21), (22).

Results of numerical experiments. Let us present the results of the solution of the test problem of control of the plate process with continuous feedback. The following values of the parameters were used in the problem:

$$a^2 = 1, \quad \lambda_0 = 0.01, \quad \lambda = 0.001, \quad \Omega = [0;1] \times [0;1], \quad t_0 = 0, \quad t_f = 1,$$

$$\mu(x) \equiv 1, \quad U(x) = 10, \quad x \in \Omega, \quad N_c = 3, \quad N_o = 4, \quad \Re_1 = \Re_2 = 1, \quad \varepsilon = 0.1,$$

$$-2 \le q_1(t) \le 20, \quad -2 \le q_2(t) \le 20, \quad -2 \le q_3(t) \le 24, \quad t \in [0;1],$$

$$-1 \le \vartheta_1(t) \le 1, \quad -1 \le \vartheta_2(t) \le 1, \quad -1 \le \vartheta_3(t) \le 1, \quad t \in [0;1],$$

$$B = [0.2; 0.4], \quad \rho_B(b) = 5(1 + \cos(10b - 3)\pi)),$$

$$\Theta = [0.8; 1.2], \quad \rho_\Theta(\theta) = 2.5(1 + \cos(5\theta - 5)\pi)),$$

$$\xi_1 \in \Omega_1 = [0.05; 0.95] \times [0.05; 0.95], \quad \xi_2 \in \Omega_2 = [0.05; 0.95] \times [0.05; 0.95],$$

$$\xi_3 \in \Omega_3 = [0.05; 0.95] \times [0.05; 0.95], \quad \xi_4 \in \Omega_4 = [0.05; 0.95] \times [0.05; 0.95],$$

$$\begin{cases} \dot{z}_{1,1}(t) = 2z_{1,2}(t), & \breve{z}_{1,1} = 0.2, \\ \dot{z}_{1,2}(t) = \vartheta_1(t), & \breve{z}_{1,2} = 0.3, \end{cases} \quad \begin{cases} \dot{z}_{2,1}(t) = -0.8 z_{2,2}(t), & \breve{z}_{2,1} = 0.75, \\ \dot{z}_{2,2}(t) = \vartheta_2(t), & \breve{z}_{2,2} = 0.83, \end{cases}$$

$$\begin{cases} \dot{z}_{3,1}(t) = -0.35 z_{3,1}(t) z_{3,2}(t) + 0.7 z_{3,2}(t), & \breve{z}_{3,1} = 0.85, \\ \dot{z}_{3,2}(t) = -0.45 z_{3,2}^2(t) + \vartheta_3(t), & \breve{z}_{3,2} = 0.67. \end{cases}$$

Let us describe the general algorithm for solving the problem of synthesizing the parameter vector y with the selected penalty coefficients $\Re = (\Re_1, \Re_2)$ and regularization parameters ε, $\hat{y}$. First, we approximate the sets B and Θ by discrete sets using the nodal points b_i and θ_j, $i = 1, 2, \ldots, N_B$,

$j = 1, 2, \ldots, N_\Theta$. When implementing the procedure (25), at each iteration with the current values of the optimized parameters y^k for the nodes values $b_i \in B$ and $\theta_j \in \Theta$, $i = 1, 2, \ldots, N_B$, $j = 1, 2, \ldots, N_\Theta$ the following steps are performed:

1) the direct initial-boundary–value problem (14), (2), (3) is solved and the Cauchy problems (15), (6) are solved;

2) the corresponding adjoint-boundary-value problem (32)–(34) and adjoint Cauchy problems (35), (36) are solved;

3) the components (27)–(31) of the gradient of the penalty functional (23), (24) are calculated;

4) one-dimensional minimization is performed in the direction of the obtained antigradient of the functional according to the procedure (26).

These steps are repeated until the stopping criterion for the functional or for the argument y is met. At the next stage, the penalty coefficients $\Re_1$, $\Re_2$ are increased by 10 times until $\Re_1 \le 10^5$, $\Re_2 \le 10^5$, the regularization parameter ε is decreased by 10 times until $\varepsilon = 0.0001$ and steps 1–4 are repeated.

To solve two-dimensional direct (14), (2), (3) and adjoint (32)–(34) loaded-boundary–value problems, a modified scheme of the variable direction method with steps in spatial variables $h_{x_1} = h_{x_2} = 0.01$ and in time $h_t = 0.01$ was used [26].

To solve the direct (15), (6) and the adjoint (35), (36) Cauchy problems, a modified scheme of the Euler method with a time step $h_t = 0.01$ was used.

To approximate the two-dimensional Dirac's $\delta(\cdot)$-function, the following everywhere smooth (differentiable) trigonometric function was used [27]:

$$\delta_\sigma(x;\eta) = \begin{cases} 0, & |x_1 - \eta_1| > \sigma_1 \quad \text{or} \quad |x_2 - \eta_2| > \sigma_2, \\ \prod_{i=1}^{2} \frac{1}{2\sigma_i}\left[1 + \cos\left(\frac{x_i - \eta_i}{\sigma_i}\pi\right)\right], & |x_1 - \eta_1| \le \sigma_1 \quad \text{and} \quad |x_2 - \eta_2| \le \sigma_2. \end{cases}$$

It is easy to verify that in this case for an arbitrary value $\sigma_i > 0$, $i = 1, 2$ the following equality holds:

$$\int_{\eta_1-\sigma_1}^{\eta_1+\sigma_1} \int_{\eta_2-\sigma_2}^{\eta_2+\sigma_2} \delta_\sigma(x;\eta)\, dx_2 dx_1 = 1.$$

In the test experiments, the values of the parameters $\sigma_i > 0$, $i = 1, 2$ of the function $\delta_\sigma(x;\eta)$ were set to $3h_{x_1}$, $3h_{x_2}$, respectively, where h_{x_1}, h_{x_2} are the steps of the grid approximation in the domain $x \in \Omega$. This choice of the type of approximation of the Dirac's $\delta(\cdot)$-function ensures sufficient smoothness of the functional $J_\Re(y)$ with respect to the optimized current locations of the measurement points ξ and the coordinates of the point heat sources $z(t)$.

Tables 1 and 2 present the results of solving the problem obtained at various iterations for two different initial points y_1^0 and y_2^0 with penalty coefficients $\Re_1 = \Re_2 = 10^5$ and regularization parameter $\varepsilon = 0.0001$. It is evident that the optimization results obtained from different starting points differ in arguments, although the difference in functionality is not significant. Here it is also necessary to take into account (as other specially conducted numerical experiments have shown) that the functional of the problem has a strong ravine structure.

TABLE 1. Results of iterations in solving the problem, obtained with the initial vector y_1^0

k	$y_1=(\alpha^1,\beta^1,\alpha^2,\beta^2,\xi)$								$J_{\Re}(y)$
0	–0.0534	–0.0231	–0.0574	–0.0823	–0.0354	–0.0875	–0.0736	–0.0137	26.982
	–0.0412	–0.0638	–0.0421	–0.0959	9.7464	9.9244	9.9124	9.8764	
	9.1374	9.8944	9.9455	9.9234	9.8154	9.9334	9.8946	9.8911	
	–0.0013	–0.0024	–0.0032	–0.0043	–0.0044	–0.0032	–0.0056	–0.0037	
	–0.0125	–0.0135	–0.0127	–0.0142	9.2345	9.7863	9.3546	9.6343	
	9.7484	9.2345	9.8765	9.7664	9.9932	9.9954	9.9945	9.9943	
	0.2200	0.2800	0.2900	0.6500	0.7300	0.7800	0.6300	0.2100	
1	–0.4938	–0.4472	–0.5177	–0.5838	–0.4622	–0.5540	–0.5820	–0.5643	0.51989
	–0.5325	–0.5346	–0.5464	–0.6443	9.7457	9.9204	9.9119	9.8788	
	9.1380	9.8994	9.9499	9.9230	9.8163	9.9357	9.8955	9.8979	
	0.0055	0.0046	0.0037	0.0032	0.0020	0.0023	0.0009	0.0031	
	–0.0083	–0.0095	–0.0086	–0.0100	9.2188	9.7696	9.3387	9.6179	
	9.7318	9.2188	9.8597	9.7498	9.9762	9.9784	9.9775	9.9773	
	0.1476	0.0642	0.4302	0.4920	0.8705	0.5991	0.7429	0.1515	
2	–0.4347	–0.4092	–0.4557	–0.5070	–0.4154	–0.5254	–0.6780	–0.7876	0.03145
	–0.6432	–0.6463	–0.3637	–0.4672	9.7598	9.9341	9.9062	9.8536	
	9.1831	9.9156	9.9764	9.9493	9.8720	9.9614	9.9213	9.9347	
	0.0047	0.0068	0.0032	0.0083	0.0028	0.0014	0.0022	0.0075	
	–0.0103	–0.0125	–0.0106	–0.0138	9.2818	9.6974	9.6578	9.7614	
	9.8732	9.3258	9.9582	9.8739	9.9814	9.9625	9.9614	9.8895	
	0.1543	0.0735	0.4760	0.4430	0.8279	0.5084	0.7906	0.2592	
3	–0.7425	–0.4884	–0.7245	–0.6872	–0.7235	–0.8231	–0.3739	–0.8127	0.00014
	–0.3573	–0.3267	–0.8426	–0.8603	9.6583	9.7813	9.8537	9.8319	
	9.5942	9.9834	9.8932	9.8932	9.9537	9.8909	9.8847	9.9993	
	0.0133	0.0105	0.0085	0.0142	0.0059	0.0043	0.0084	0.0126	
	–0.0084	–0.0095	–0.0074	–0.0094	9.8437	9.9429	9.8320	9.8398	
	9.6228	9.5128	9.9328	9.9337	9.9388	9.9294	9.9445	9.9825	
	0.1827	0.1489	0.5860	0.4925	0.6384	0.7329	0.8525	0.3592	

TABLE 2. Results of iterations in solving the problem, obtained with the initial vector y_2^0

k	$y_2=(\alpha^1,\beta^1,\alpha^2,\beta^2,\xi)$								$J_{\Re}(y)$
0	–0.1125	–0.1386	–0.1089	–0.1345	–0.1257	–0.1467	–0.1908	–0.1874	4.3345
	–0.1866	–0.1523	–0.1523	–0.1784	8.9345	9.4531	9.8756	9.7463	
	9.4728	9.7863	9.8365	9.6643	9.9439	9.5549	9.7594	9.9043	
	–0.0061	–0.0118	–0.0127	–0.0105	0.0114	0.0119	0.0127	0.0139	
	–0.0145	–0.0142	–0.0175	–0.0147	9.9372	9.9534	9.8836	9.9433	
	9.4548	9.8649	9.7565	9.9460	9.9354	9.7395	9.9574	9.8643	
	0.6524	0.2127	0.8325	0.8814	0.3125	0.6838	0.2715	0.3290	
1	–0.5180	–0.5308	–0.4573	–0.5490	–0.5832	–0.5725	–0.5390	–0.5983	0.58196
	–0.6870	–0.5562	–0.4930	–0.6109	8.9404	9.4611	9.8807	9.7538	
	9.4799	9.7952	9.8294	9.6770	9.9563	9.5644	9.7688	9.9160	
	0.0074	0.0021	–0.0032	0.0011	0.0325	0.0319	0.0284	0.0339	
	–0.0235	–0.0149	–0.0128	–0.0142	9.9324	9.9486	9.8789	9.9385	
	9.4503	9.8602	9.8429	9.9413	9.9306	9.8548	9.8526	9.8596	
	0.6890	0.1553	0.9063	0.8064	0.3512	0.5799	0.2990	0.1719	
2	–0.4961	–0.3827	–0.4223	–0.4357	–0.3933	–0.7810	–0.3381	–0.4573	0.03259
	–0.5401	–0.4536	–0.2277	–0.8934	9.9432	9.9466	9.9245	9.8392	
	9.3439	9.5487	9.9632	9.5421	9.9363	9.5703	9.8821	9.9559	
	0.0094	0.0028	–0.0048	0.0038	0.0251	0.0271	0.0337	0.0255	
	–0.0112	–0.0128	–0.0137	–0.0132	9.9256	9.9248	9.8521	9.9727	
	9.4029	9.8535	9.9452	9.8945	9.8838	9.9281	9.9258	9.9449	
	0.7224	0.1426	0.8834	0.8301	0.3857	0.5481	0.3265	0.1255	
3	–0.5457	–0.5523	–0.4719	–0.5752	–0.6027	–0.5804	–0.5314	–0.6017	0.00013
	–0.7399	–0.5825	–0.5075	–0.6430	8.9462	9.4668	9.8845	9.7598	
	9.4849	9.7997	9.8532	9.6821	9.9650	9.5703	9.7731	9.9229	
	0.0122	0.0036	–0.0032	0.0011	0.0384	0.0374	0.0339	0.0397	
	–0.0032	–0.0048	–0.0097	–0.0050	9.9256	9.9418	9.8721	9.9317	
	9.4439	9.8535	9.7452	9.9345	9.9238	9.7281	9.9458	9.8528	
	0.7515	0.1268	0.9461	0.8799	0.4178	0.4880	0.3767	0.0655	

Fig. 1 and 2 show the graphs of the trajectory of heat sources and the corresponding controls for $t\in[0;1]$ for two different initial values of the parameter vector y_1^0, y_2^0 and for the obtained optimal vectors y_1^*, y_2^*, with penalty coefficients $\Re_1=\Re_2=10^5$ and regularization parameter $\varepsilon=0.0001$.

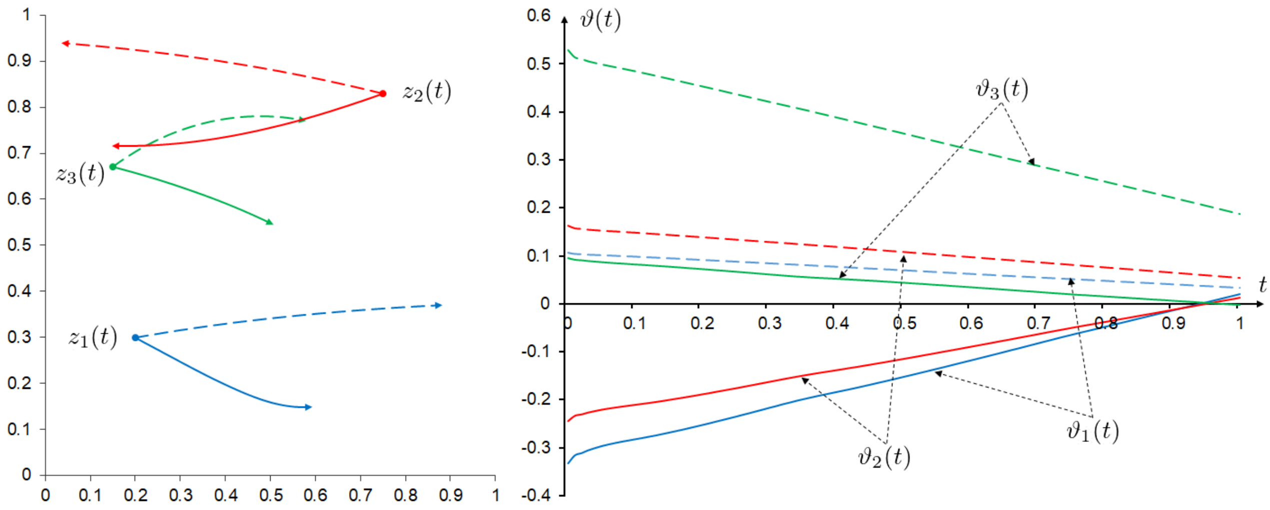

FIG. 1. Graphs of heat source trajectories and corresponding controls for the initial value of the vector y_1^0 (dashed lines) and for the obtained optimal vector y_1^* (solid lines)

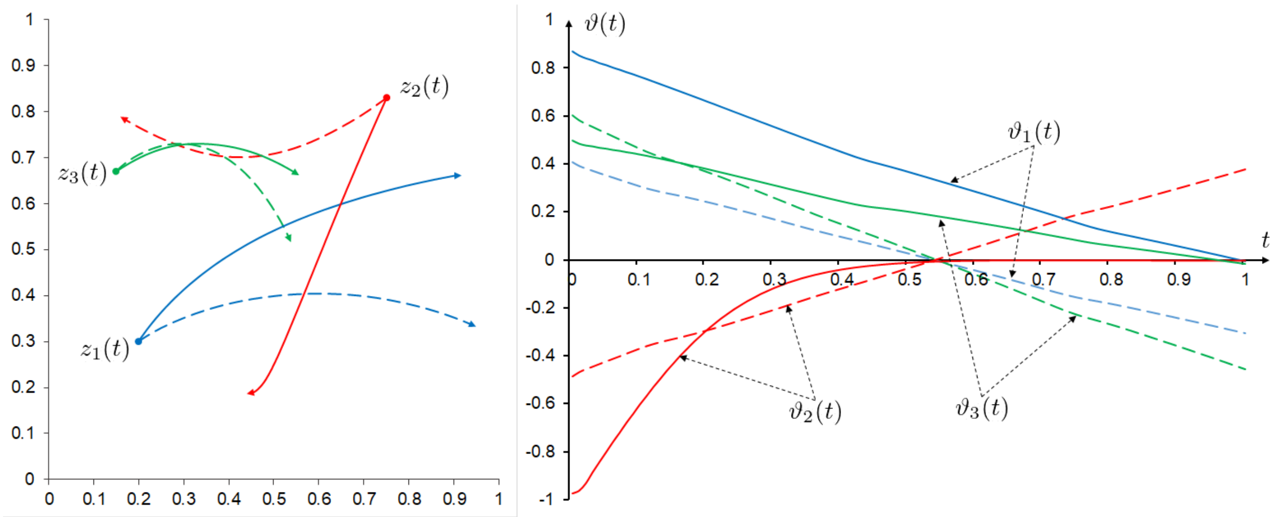

FIG. 2. Graphs of heat source trajectories and corresponding controls for the initial value of the vector y_2^0 (dashed lines) and for the obtained optimal vector y_2^* (solid lines)

Fig. 3 shows the graphs of the power of heat sources for the synthesized optimal vector of feedback parameters y_1, y_2.

Fig. 4 shows the graphs of the function

$$\Phi(t;y^*)=\int_B\int_\Theta\left\{\iint_\Omega \mu(x)\left[u(x,t;y^*,b,\theta)-U(x)\right]^2 dx\right\}\rho_\Theta(\theta)\rho_B(b)d\theta db\,,$$

where $u(x,t;y^*,b,\theta)$ is the solution to the initial-boundary–value problem (14), (2), (3) with the optimal vector of feedback parameters y^* obtained from the numerical solution of the problem.

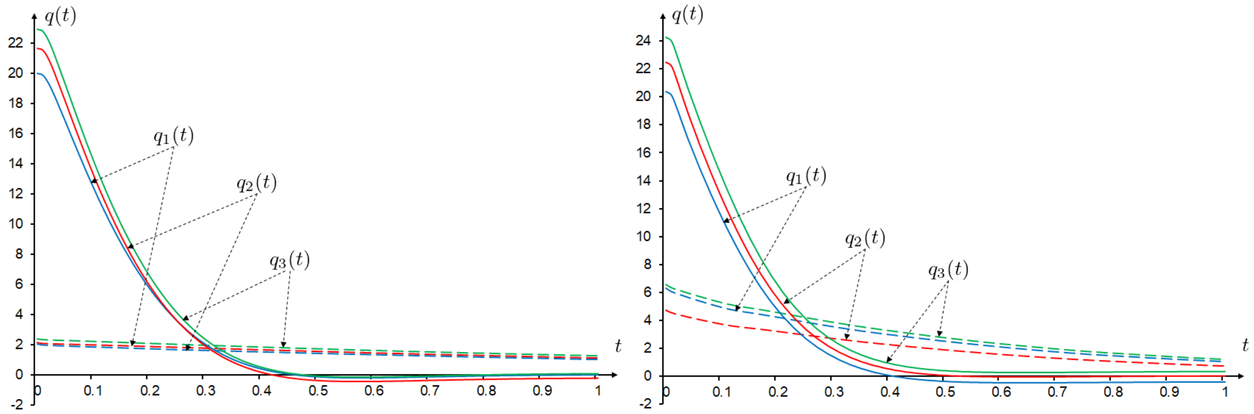

FIG. 3. Graphs of the power of heat sources $q(t)$ over time $t \in [0; 1]$ for two initial values of the parameter vector y_1^0, y_2^0 and for the obtained vectors y_1^*, y_2^*

From the graphs shown in Fig. 4 one can see how the temperature distribution on the plate approached the desired distribution over time.

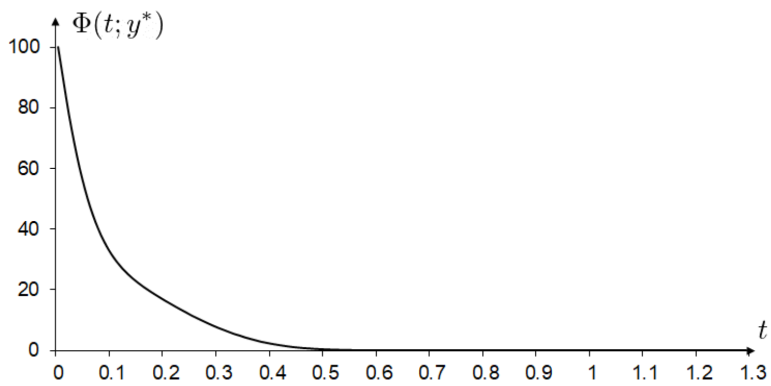

FIG. 4. Graphs of the values of the function $\Phi(t;y^*)$ in $t \in [0; 1.3]$ for the feedback parameters y_1^*

Computer experiments were conducted to control the plate heating process with optimal values of synthesized feedback parameters under the assumption that measurements are carried out with errors (interference), namely:

$$\breve{u}_j^{\chi}(t) = u(\xi_j, t)\left[1 + \chi_j(t)\right], \quad t \in [t_0, t_f], \quad i = 1, 2, \ldots, N_c.$$

Here $\chi_j(t)$ for each t is a random variable uniformly distributed on the segment $[-\zeta, \zeta]$, ζ is determines the maximum possible level of error. In the experiments conducted, the values of ζ were chosen equal to 0.01, 0.03, 0.05, which corresponds to a measurement error of 1%, 3% and 5% from the exact (calculated) values of the measured quantities.

An important indicator of the quality of control of the heating process with the used feedback parameters is the function $\Phi(t; y^*)$, which numerically characterizes the result of process control on average for all possible values of external influences. Fig. 5 shows the graphs of the function $\Phi(t; y^*)$ obtained with optimal feedback parameters and with measurement error levels equal to 0% (no error), 1%, 3% and 5%. These graphs show that the quality of control of the stabilization process corresponds to the magnitude of the error in the measurements of the process state. As can be seen from Fig. 5, the function $\Phi(t; y^*)$, and therefore the heating process itself, is quite stable against measurement errors, and this stability is maintained when the process is controlled at $t \geq 1$. In Fig. 5,*a* due to the small scale of the value of the function $\Phi(t; y^*)$ at $t \in [0; 0.6]$ at different error levels, it is practically the same. Therefore, in Fig. 5,*b* for $t \in [0.6; 1.3]$ the scale for the axis of function values is increased.

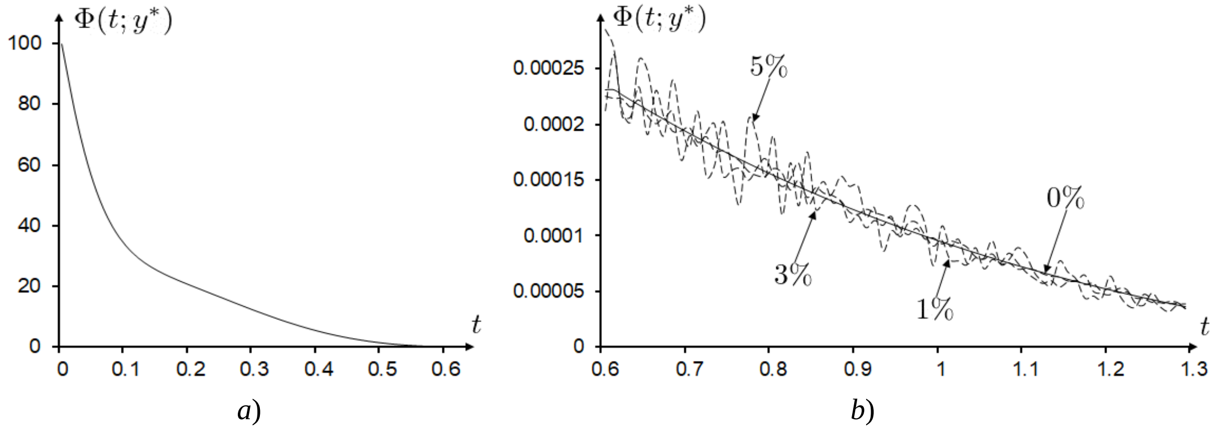

FIG. 5. Graphs of the values of the function $\Phi(t; y^*)$ for the feedback parameters y_1^* at the error level of 1 %, 3 %, 5 %: *a* – $t \in [0;\ 0.6]$; *b* – $t \in [0.6;\ 1.3]$

Conclusion. An approach to feedback control of the motion and power of lumped sources in systems with distributed parameters is proposed. The problem of controlling moving heat sources used to heat a plate is considered. The powers and control actions on the motion of point sources are determined in the form of proposed dependencies on the results of the taken measurements. The differentiability of the functional with respect to the feedback parameters is shown, and formulas for the gradient of the functional with respect to the synthesized parameters are obtained. The formulas allow us to use efficient numerical methods of first-order optimization and available standard software packages to solve the problem of synthesis of control of lumped sources.

The proposed approach to control of lumped sources with feedback can be used in automatic control and regulation systems of lumped sources for many other technological processes and technical objects.

Funding. The author did not receive any funding for conducting research or writing the article.

References

1. Lions J.-L. Controle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles, Paris: Dunod Ganthier-Villars, 1969.
2. Butkovskii A.G. Control methods for systems with distributed parameters. M.: Nauka, 1975. (in Russian)
3. Butkovskii A.G., Pustylnikov L.M. Theory of mobile control of distributed parameter systems. M.: Nauka, 1980. (in Russian)
4. Polyak B.T., Khlebnikov M.V., Rapoport L.B. Mathematical theory of automatic control. M.: LENAND, 2019. 504 p. (in Russian)
5. Egorov A.I. Fundamentals of the control theory. M.: Fizmatlit, 2004. (in Russian)
6. Sirazetdinov T.K. Optimization of distributed parameter systems. M.: Nauka, 1977. (in Russian)
7. Utkin V.I., Sliding conditions in optimization and control problems. M.: Nauka, 1981. (in Russian)
8. Ray W.H. Advanced Process Control. McGraw-Hill Book Company, 1980. 376 p.
9. Emelyanov S.V., Korovin S.K. New types of feedback. M.: Nauka, 1997. (in Russian)
10. Guliyev S.Z. Synthesis of zonal controls for a problem of heating with delay under nonseparated boundary conditions, *Cybern. Syst. Analysis.* 2018. **54** (1). P. 110–121. https://doi.org/10.1007/s10559-018-0012-5
11. Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F. The mathematical theory of optimal processes. M.: Nauka, 1983. 393 p. (in Russian)
12. Bellman R., Gliksberg I., Gross O. Some problems of the mathematical theory of control processes. M.: Foreign Literature Press, 1962. 335 p. (in Russian)
13. Letov A.M. Analytical Design of Controllers. *I–III, Avtom. Telemekh.* 1960. **21** (4). P. 436–441; 1960. **21** (5). P. 562–568; 1961. **21** (6). P. 661–665; 1961. **22** (4). P. 425–435; 1962, **23** (11). P. 1405–1413.
14. Krasovskiy N.N. Theory of control of motion. M.: Nauka, 1968. 476 p. (in Russian)
15. Polyak B.T., Shcherbakov P.S. Robust Stability and Control. M.: Nauka, 2002. (in Russian)
16. Sergienko I.V., Deineka V.S. Optimal control of distributed systems with conjugation conditions. New York: Kluwer Acad. Publ, 2005.
17. Tikhonov A.N., Samarskii A.A. Equations of Mathematical Physics. New York: Dover, 2011.
18. Lions J.-L., Magenes E. Problèmes aux limites non homogènes at application. Vol. 1. Paris, 1968.
19. Vasiliev F.P. Optimization methods. M.: Factorial Press, 2002. (in Russian)
20. Mardanov M.J., Sharifov Y.A., Zeynalli M.F. Existence and uniqueness of the solutions to impulsive nonlinear integro-differential equations with nonlocal boundary conditions. *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* 2019. **45** (2). P. 222–233. https://doi.org/10.29228/proc.6
21. Nakhushev A.M. Loaded equations and their applications. M.: Nauka, 2012. (in Russian)
22. Alikhanov A.A., Berezgov A.M., Shkhanukov-Lafishev M.X. Boundary value problems for certain classes of loaded differential equations and solving them by finite difference methods. *Comput. Math. and Math. Phys.* 2008. **48**. P. 1581–1590. https://doi.org/10.1134/S096554250809008X
23. Abdullaev V.M., Aida-zade K.R. Numerical method of solution to loaded nonlocal boundary value problems for ordinary differential equations. *Comput. Math. Math. Physics.* 2014. **54** (7). P. 1096–1109. https://doi.org/10.1134/S0965542514070021
24. Abdullayev V.M., Aida-zade K.R. Finite-difference methods for solving loaded parabolic equations. *Comp. Math. Math. Phys.* 2016. **56** (1). P. 93–105. https://doi.org/10.1134/S0965542516010036
25. Pshenichnyi B.N. Necessary Conditions for an Extremum. M.: Nauka, 1969. 152 p. (in Russian)
26. Samarskii A.A. Theory of Difference Schemes. M.: Nauka, 1989. (in Russian)
27. Aida-zade K.R., Bagirov A.G. On the problem of spacing of oil wells and control of their production rates. *Autom. Remote Control.* 2006. **67 (**1). P. 44–53. https://doi.org/10.1134/S0005117906010024

Received 17.07.2025

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УДК 519.6

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Синтез керування потужністю та рухом джерел нагріву пластин та оптимізація розміщення точок вимірювання температури

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Вступ. Розглянута проблема належить до задач оптимального керування зосередженими джерелами в системах з розподіленими параметрами. Такі задачі описуються початково-крайовими задачами відносно рівнянь з частинними похідними різних типів. Теорія задач оптимального керування для систем з розподіленими параметрами почала найактивніше розвиватися в 60-х роках минулого століття. Це було пов'язано з необхідністю вирішення таких важливих процесів, як розробка великих нафтогазових родовищ та трубопровідний транспорт вуглеводневої сировини. Подібні задачі актуальні в металургії, екології та багатьох інших галузях промисловості.

Мета роботи. У цій статті досліджується проблема синтезу оптимального керування рухомими точковими джерелами тепла для нагріву двовимірної пластини. Оптимізовано потужності джерел та їх траєкторії руху, що описуються звичайними диференціальними рівняннями. Крім того, у розглянутій задачі також оптимізовано розташування точок вимірювання температури. Отримано необхідні умови оптимальності для параметрів зворотного зв'язку та координати для встановлення точок вимірювання. Умови містять формули для компонентів градієнта цільового функціонала відповідно до оптимізованих параметрів.

Результати. Отримано необхідні умови оптимальності синтезованих параметрів зворотного зв'язку та формули для компонентів градієнта цільового функціонала від цих параметрів. Ці формули дозволяють використовувати ефективні методи числової оптимізації першого порядку для вирішення задачі синтезу параметрів зворотного зв'язку. Представлено результати комп'ютерних експериментів, отриманих за допомогою методів числової оптимізації першого порядку.

Висновки. Запропоновано підхід до керування зі зворотнім зв'язком рухом та потужністю зосереджених джерел у системах з розподіленими параметрами. Розглянуто задачу керування рухомими джерелами тепла, що використовуються для нагрівання пластини. Потужності та керуючі впливи на рух точкових джерел визначаються у вигляді запропонованих залежностей від результатів проведених вимірювань. Показано диференційованість функціонала відносно параметрів зворотного зв'язку та отримано формули для градієнта функціонала відносно синтезованих параметрів. Формули дозволяють використовувати ефективні числові методи оптимізації першого порядку та доступні стандартні пакети програмного забезпечення для вирішення задачі синтезу керування зосередженими джерелами.

Ключові слова: нагрівання пластини, керування зі зворотним зв'язком, рухомі джерела, точки вимірювання температури, параметри зворотного зв'язку.