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REGRET FUNCTION MINIMIZATION ALGORITHMS

The problem of making optimal decisions under uncertainty is examined through the minimization of Savage's regret function. Two algorithms are developed based on the subgradient projection method with automatic step-size adjustment, adapted for high-dimensional decision-making environments. Unlike traditional methods, the proposed approaches evaluate only selected subsets of states of nature and constraints at each iteration, significantly reducing computational cost. Theoretical convergence of the algorithms to the optimal decision set is proven, despite the absence of exact regret function values. These methods provide a practical and scalable solution for regret-based decision-making models under uncertainty.

***Keywords**: uncertainty, decision-making, Savage function, regret minimization, convex optimization, subgradient method.*

Introduction. Most socio-economic systems face uncertainty in their operation, more specifically, in the process of developing and implementing a decision. Major attention must be paid to the analysis of the states of nature when assessing the degree of their influence on the evolution of these systems. The same decision could be "good" with respect to one state of nature and "bad" with respect to another. The classification into "good" or "bad" naturally depends on the adopted criterion. To date, several types of decision criteria used under conditions of uncertainty are known [1]. One of these is Savage's criterion [2], which is usually used when the decision-maker reacts "painfully" to large losses. That is, when they cannot accept the gap between the actual the expected costs. Thus, the use of Savage's criterion somehow "involves" the decision-maker, selecting a decision that would be a "compromise" for all states of nature. Therefore, according to this criterion, the maximum deviation function is defined in relation to all the states of nature that could occur in the given decision-making process.

In this work, as well as in the publication [3], two decision-making models conditioned by the uncertainty of the states of nature are considered. Both models express Savage's viewpoint, which consists of determining a decision that would ensure the achievement of the minimum value of the maximal regret. In [3], the addressed problems are solved using the subgradient projection method [4, 5], taking into account all states of nature at each iteration. In the case with general constraints, it is solved using all the functions from these constraints in the calculations.

In the present article, similar methods are proposed, except that at each iteration, operations are performed with a specific, much smaller subset of the total number of states of nature. Additionally, in the model with general constraints, only a few of them are taken into account. These modifications can significantly reduce the volume of calculations at each iteration, which can streamline the entire process of approximate problem-solving in question.

The considered models have the following form:

Model I:

$$R_s(u) \underset{u\in U\subset E^n}{\rightarrow} \min, \quad (1)$$

Model II:

$$R_s(u) \to \min_{u \in D=\{u \in U: F(u) \le 0\}}, \tag{2}$$

here:

$R_s(u) = \max_{i \in I}\{\bar{r}(u,\ q_i)\}$ – the Savage function;

$\bar{r}(u,\ q_i) = r(u,\ q_i) - r_i^*$ – the Regret function, determined by the state of nature $q_i \in Q = \{q_i, \dots, q_m\}$;

$r_i^* = r\left(u_{(i)}^*, q_i\right) = \min r(u, q_i)$ and $u \in U$ in model I, $u \in D$ in model II;

$F(u) = \max_{j \in J}\{F_j(u)\}$.

It is assumed that all functions $r(u,\ q_i), i \in I = \{1, \dots, m\}$ and $F_j(u), j \in J = \{1, \dots, t\}$ are continuous and convex on the compact and convex U in the euclidian space E^n.

Remark. In order to be able to work with the function $\bar{r}(u,\ q_i)$ it is required to know its minimum value r_i^* which is realistically impossible. When implementing the algorithms, in each iteration, certain estimations of r_i^*, will be used instead of its actual value.

In the following we will consider certain partitions of the sets I and J into subsets $I_1, \dots, I_M$ și $J_1, \dots, J_N$, respectively, such that:

$$\bigcup_{i=1}^{M} I_i = I, \qquad \bigcup_{j=1}^{N} J_j = J,$$

$$I_{i_1} \cap I_{i_2} = \emptyset, i_1 \ne i_2, \qquad J_{j_1} \cap J_{j_2} = \emptyset, j_1 \ne j_2,$$

$$I_i \ne \emptyset, i = \overline{1, M}. \qquad J_j \ne \emptyset, j = \overline{1, N}.$$

Description of algorithm I (for solving problem (1)).

Suppose that at iteration k the points $u_{(i)}^k, i = \overline{1,m}$ and u^k are already known. In order to determine the following set of points $u_{(i)}^{k+1}, i = \overline{1,m}$ and u^{k+1} the following steps are performed:

$$u_{(i)}^{k+1} = \begin{cases} P_U(u_{(i)}^k - h_{(i)k} \cdot \eta_{(i)}^k), & if\ \ i \in \tilde{I}_{i_k} \\ u_{(i)}^k, & if\ \ i \notin \tilde{I}_{i_k}, k = 0,1, \dots \end{cases} \tag{3}$$

$$u^{k+1} = P_U(u^k - h_k \cdot \eta^k), k = 0,1, \dots \tag{4}$$

where:

$i_k = k - M \cdot \left[\frac{k}{M}\right] + 1$, (obviously, the index i_k takes the consecutive values $0,1, \dots, M$);

$\left[\frac{k}{M}\right]$ – the integer part of the ratio ${}^{k}/_{M}$;

$\tilde{I}_{i_k} = I_{i_k} \cup \{i^{k-1}\}$, ($i^{-1}$ any element from I);

$$\eta_{(i)}^k = \begin{cases} gr(u_{(i)}^k, q_i)/\left\|gr(u_{(i)}^k, q_i)\right\|, & if\ \ gr(u_{(i)}^k, q_i) \ne 0 \\ 0, & otherwise \end{cases}, \tag{5}$$

where: $gr(u_{(i)}^k, q_i)$ – is an arbitrary subgradient of the function $r(u_{(i)}, q_i)$, computed at the point $u_{(i)} = u_{(i)}^k$, and $\left\|gr(u_{(i)}^k, q_i)\right\|$ denoted the magnitude (the Euclidian norm) of this subgradient.

The vector η^k from (4) is determined in the following way:

$$\eta^k = \begin{cases} g\tilde{R}^k(u^k)/\left\|g\tilde{R}^k\left(u^k\right)\right\|, & for\ g\tilde{R}^k(u^k) \ne 0, \\ 0, otherwise. \end{cases} \tag{6}$$

$g\tilde{R}^k(u^k)$ – an arbitrary subgradient of the function:

$$\tilde{R}^k(u) = \max_{i\in\tilde{I}_{i_k}} \{r(u,\ q_i) - r(u_{(i)}^k, q_i)\}, \tag{7}$$

computed at the point $u = u^k$. The value i^k represents the index i from $\tilde{I}_{i_k}$ for which the value $\tilde{R}^k(u^k)$ is attained (if there are multiple such indices, one is chosen arbitrarily).

Remark. Numbers i^k and i_k are not to be confused. Numbers $h_{(i)k}$ and h_k represent the step sizes for minimising the functions $r(u,\ q_i)$ and $R_s(u)$, respectively, at iteration k.

It will be assumed henceforth that the sequences $h_{(i)k}, i = \overline{1,m}$ and h_k satisfy the conditions:

$$\begin{gathered} h_{(i)k}, h_k \geq 0;\ h_{(i)k}, h_k \to 0; \\ \textstyle\sum_{k=0}^{\infty} h_{(i)k\cdot M+l} = \infty, l = 0,1,\dots,M-1; \sum_{k=0}^{\infty} h_k = \infty\,. \end{gathered} \tag{8}$$

Remark. For the M relations in (8) (for $l = \overline{0, M-1}$) to hold, it is necessary that all the subsequences $\{h_{(i)k\cdot M+l}\}_{k=0}^{\infty}$, for $l = 0,1,\dots,M-1$, form divergent series.

Constructing such sequences is not difficult. For example, if a sequence $\{\tilde{h}_{(i)l}\}$ satisfies the conditions:

$$\tilde{h}_{(i)l} \geq 0,\ \tilde{h}_{(i)l} \to 0,\ \textstyle\sum_{l=0}^{\infty} \tilde{h}_{(i)l} = \infty,$$

and the sequence $\{h_{(i)k}\}$ is to be defined in the following way:

$$h_{(i)k} = \tilde{h}_{(i)l}, \text{ for } l\cdot M \leq k \leq (l+1)M - 1,$$

then the sequence $\{h_{(i)k}\}$ has all of its subsequences in the form $\{h_{(i)k\cdot M+l}\}$, $l = 0,1,\dots,M-1$, with the same properties as the sequence $\{\tilde{h}_{(i)l}\}$.

Theorem 1. Let U^* be the set of minimum points of problem (1). Under the conditions that requirements (3)–(8) are satisfied, the following statements hold:

$$\lim_{k\to\infty} \min_{u^*\in U^*} \|u^k - u^*\| = 0\,, \lim_{k\to\infty} \tilde{R}^k(u^k) = R_S(u^*). \tag{9}$$

Remark. The justification of the relations (9), with some modifications, is based almost entirely on the proof of *Theorem 1* in the work [3]. However, unlike [3], at each iteration k, the subgradient projection method is not applied with respect to all states of nature, whose number can be quite large, but only with respect to the states of nature corresponding to the subset of indices I_{i_k} (one of the subsequences $I_1, I_2, \dots, I_M$) and the "representative" index, denoted i^{k-1}, from the previous iteration $(k-1)$. Similarly, with respect to these mentioned states, the value of the Savage function $R_S(u)$ is evaluated through the estimate $\tilde{R}^k(u)$, defined in (7). The requirement $\sum_{k=0}^{\infty} h_{(i)k\cdot M+l} = \infty, l = \overline{0, M-1}$, from (8), is necessary to ensure the convergence, with respect to each state of nature $q_i \in Q$, of the sequences $\{u_{(i)}^k\}, i \in I$, to the solutions $u_{(i)}^*$ and respectively, the value sequence $\{r(u_{(i)}^k, q_i)\}, i \in I$, to the minimal values r_i^*, regardless of the chosen start points $u_{(i)}^0, i \in I$.

***Description of Algorithm II. Solving Problem* (2).**

As in the application of Algorithm I, the transition rules (3) and (4) are used to move from iteration (k) to the following iteration $(k+1)$. However, in this case, the method for determining the vectors $\eta_{(i)}^k$ and η^kdiffers. Specifically:

$$\eta_{(i)}^k = \begin{cases} gr(u_{(i)}^k, q_i)/\|gr(u_{(i)}^k, q_i)\|\,, if\ \tilde{F}_{(i)}^k(u_{(i)}^k) \leq \varepsilon_{(i)k}\ and\ gr(u_{(i)}^k, q_i) \neq 0, \\ g\tilde{F}_{(i)}^k(u_{(i)}^k)/\|g\tilde{F}_{(i)}^k(u_{(i)}^k)\|, if\ \tilde{F}_{(i)}^k(u_{(i)}^k) > \varepsilon_{(i)k}\ and\ g\tilde{F}_{(i)}^k(u_{(i)}^k) \neq 0, \\ \quad 0, otherwise. \end{cases} \tag{10}$$

Here:

$$i \in \tilde{I}_{i_k};$$
$$\tilde{F}_{(i)}^k(u) = \max_j\{F_j(u)\}, j \in \tilde{J}_{j_k}^i; \quad (11)$$
$$\tilde{J}_{j_k}^i = \tilde{J}_{j_k} \cup \{j_i^{k-1}\};$$

j_i^{-1} – is any element from set J, $i = \overline{1,m}$;

$\varepsilon_{(i)k}$ – is the ceiling (tolerance threshold) with respect to the state of naturei q_i, $i \in \tilde{I}_{i_k}$ at iteration k.

$$j_k = k - N \cdot \left[\frac{k}{N}\right] + 1, k = 0,1, \ldots$$

$$\eta^k = \begin{cases} g\tilde{R}^k(u^k)/\|g\tilde{R}^k(u^k)\|, if\ \tilde{F}^k(u^k) \le \varepsilon^k\ și\ g\tilde{R}^k(u^k) \ne 0, \\ g\tilde{F}^k(u^k)/\|g\tilde{F}^k(u^k)\|, if\ \tilde{F}^k(u^k) > \varepsilon^k\ și\ g\tilde{F}^k(u^k) \ne 0, \\ 0, otherwise. \end{cases} \quad (12)$$

In (12) vectors $g\tilde{R}^k(u)$ și $g\tilde{F}^k(u)$ are arbitrary subgradients of the function $\tilde{R}^k(u)$ and $\tilde{F}^k(u)$, respectively, computed at point $u = u^k$. $\tilde{R}^k(u)$ is defined in (7), nd $\tilde{F}^k(u) = \max_j\{F_j(u)\}$ with respect to all values $j \in \tilde{J}_{j_k} = \tilde{J}_{j_k} \cup \{j^{k-1}\}$; j^{-1} – being an arbitrary element from the set J.

Next, it is necessary to identify the requirements regarding the parameters $h_{(i)k}$, h_k, $\varepsilon_{(i)k}$, ε_k. The following conditions will be assumed:

$$h_{(i)k}, h_k \ge 0;\ \varepsilon_{(i)k}, \varepsilon_k > 0;\ h_{(i)k}, h_k, \varepsilon_{(i)k}, \varepsilon_k \to 0;$$
$$h_{(i)k}/\varepsilon_{(i)k},\ h_k/\varepsilon_k \to 0; \quad (13)$$

$$\sum_{k=0}^{\infty} h_{(i)k\cdot M+l} \cdot \varepsilon_{(i)k\cdot M+l} = \infty, l = 0,1, \ldots, \mathrm{M} - 1,$$
$$\sum_{k=0}^{\infty} h_k \cdot \varepsilon_k = \infty. \quad (14)$$

The following statement holds:

Theorem 2. Let U^* be the set of minimum points of problem (2). If the algorithm defined by rules (3) and (4) is applied, with the additional conditions (10)–(14) imposed, then, with respect to problem (2), the equalities (9) hold.

The proof of this theorem is based on the reasoning presented in Theorem 3 of [3] and on the results related to minimax-type problems from the work [6]. At the same time, for a clearer understanding of how the algorithm unfolds, as well as its convergence toward the optimal decision domain, the following remark is considered necessary.

Remark. At the first M iterations ($k = \overline{0, M-1}$) index i (which represents the states of nature) goes through all the values from $I_1 \cup \{i^{k-1}\}$ (for $k = 0$), from $I_2 \cup \{i^{k-1}\}$ (for $k = 1$) and so on, from $I_M \cup \{i^{k-1}\}$ (for $k = M - 1$), and index j (which is associated with restriction j or with the function $F_j(u)$) goes through all the values from the set $J_1 \cup \{j_i^{k-1}\}$ consecutively to determine $\eta_{(i)}^k$ from (10) ($i = \overline{1,m}$) and goes through all the values from the set $J_1 \cup \{j^{k-1}\}$, to determine η^k from (12).

For the following M iterations ($k = \overline{M, 2M-1}$) i takes consecutive values from the set $I_1 \cup \{i^{k-1}\}$ (for $k = M$), ..., from $I_M \cup \{i^{k-1}\}$ (for $k = 2M - 1$), and index j from $J_2 \cup \{j_i^{k-1}\}$ – to define (10) and, respectively, from $J_2 \cup \{j^{k-1}\}$ – to define(12). And so on, for iterations $k = \overline{(N-1)\cdot M, N\cdot M - 1}$, similarly, index i, takes values from the set $I_1 \cup \{i^{k-1}\}$, $I_2 \cup \{i^{k-1}\}$,..., $I_M \cup \{i^{k-1}\}$, whereas index j – values from $J_N \cup \{j_i^{k-1}\}$, in order to determine the vector $\eta_{(i)}^k$ from (10) and values from $J_1 \cup \{j^{k-1}\}$ to determine vector η^k from (12). After $N \cdot M$ iterations, everything repeats except the numbers i^{k-1}, j_i^{k-1} and j^k can vary from iteration to iteration.

Conclusion. The algorithms proposed in this paper were developed in order to minimise the Savage function, which is a special function, especially because it is impossible to compute its exact values and the values of its subgradients. Despite these difficulties, during the execution of the algorithms, estimations of any desired a priori precision can be obtained for both the values of the Savage function and its subgradients. This makes it possible (under certain conditions regarding the step size adjustments, and in more general cases, also concerning the tolerance threshold values for constraint violations) to ensure convergence of the algorithms toward the set of optimal decision alternatives.

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References

1. Hamdy A. T. Operations Research. Collier Macmillan Publishers, London ,1982.
2. Savage L.J. The theory of statistical decision. *J. Amer. Statist. Assoc*. 1951. **46** (1). P. 55–67. https://doi.org/10.1080/01621459.1951.10500768
3. Godonoaga A.F., Blanutsa S.A., Chumakov B.M. Methods for minimizing the Savage function with various constraints. *Cybernetics and Computer Technologies*. 2024. 1. P. 18–26. (in Ukrainian) https://doi.org/10.34229/2707-451X.24.1.2
4. Shor N.Z. Nondifferentiable optimization and polynomial problems. Boston, Kluwer Academic Publishers, 1998. https://doi.org/10.1007/978-1-4757-6015-6
5. Yermolyev Yu.M. Methods of stochastic programming. M.: Nauka, 1976. 240 p. (in Russian)
6. Godonoaga A.F., Chumakov B.M. Deterministic and stochastic schemes of the subgradient projection method. *Theory of optimal solutions*. 2015. 14. P. 90–97. (in Russian) http://jnas.nbuv.gov.ua/article/UJRN-0000475038

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Алгоритми мінімізації функції жалю

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Вступ. У статті розглянуто задачу прийняття оптимальних рішень в умовах невизначеності шляхом мінімізації функції жалю за Севіджем. Ця функція дозволяє оцінити відхилення між реальним результатом і найкращим можливим результатом при кожному можливому стані природи. Такий підхід є особливо корисним у ситуаціях, коли особа, що приймає рішення, є схильною до уникнення великих втрат і прагне до компромісного рішення, яке забезпечує найменший максимальний жаль.

Запропоновано два ефективні чисельні алгоритми, побудовані на основі методу проєкцій субградієнта. Особливістю цих алгоритмів є те, що на кожній ітерації розглядається лише обмежена кількість станів природи та обмежень, що суттєво зменшує обчислювальні витрати в порівнянні з класичними підходами. У першій моделі розглядаються задачі з невизначеністю без складних обмежень, а друга модель узагальнює підхід на випадки з великою кількістю загальних обмежень. Для обох алгоритмів доведено збіжність до оптимального рішення за певних умов на послідовності кроків та вибору підмножин.

Також у статті зазначено, що під час виконання алгоритмів не обов'язково точно обчислювати значення функції жалю та її субградієнтів – замість цього використовуються апроксимації, які збігаються до істинних значень. Це робить запропоновані методи придатними до практичного використання в складних реальних умовах, де точне моделювання всіх можливих станів природи є неможливим або економічно недоцільним.

У подальших дослідженнях планується застосування запропонованих алгоритмів до реальних задач у галузях економіки, логістики та операційних досліджень, де проблема прийняття рішень в умовах невизначеності є особливо актуальною.

Мета роботи. Розробити та обґрунтувати ефективні алгоритми для мінімізації функції жалю у процесах прийняття рішень в умовах невизначеності, з урахуванням зменшення обчислювальної складності за рахунок роботи з підмножинами станів природи та обмежень.

Результати. Подано два алгоритми, що базуються на модифікованому методі субградієнтів. Перший алгоритм орієнтований на задачі без складних обмежень, другий – на загальні обмеження. Доведено збіжність обох методів до оптимального рішення. Зменшення кількості розрахунків на кожній ітерації досягається шляхом роботи лише з частиною даних, що дозволяє зберегти ефективність навіть при великій кількості станів.

Висновки. Запропоновані алгоритми забезпечують ефективне вирішення задач прийняття рішень з урахуванням ризику та невизначеності. Їх застосування особливо доцільне в задачах великого масштабу, де традиційні методи є занадто витратними. Алгоритми дозволяють апроксимувати значення функції жалю та її субградієнтів з достатньою точністю для забезпечення збіжності.

Ключові слова: прийняття рішень, невизначеність, мінімізація жалю, функція Севіджа, метод субградієнта, опукла оптимізація.