

ON THE SOLUTION OF THE TRAVELING SALESMAN PROBLEM BY A MODIFICATION OF THE HUNGARIAN METHOD

Introduction. We consider the traveling salesman problem of visiting n specified cities, visiting each one only once and returning to the point of departure, while minimizing the cost of the path f . Let C_{ij} be the cost of the transition from city i to city j . The problem is to find a permutation of integers from 1 to n that minimizes the cost $f = C_{i_1 i_2} + C_{i_2 i_3} + C_{i_3 i_4} \dots + C_{i_n i_1}$ of the route.

Numerous exact and approximate algorithms have been developed to solve the traveling salesman problem [1–6]. Among the exact solution methods are the exhaustive search method, dynamic programming, branch and bound methods, and branch-and-cut algorithm. Instead of exact algorithms, which are often impractical for large-scale problems, approximate algorithms are used, which find solutions close to the optimal one in an acceptable time. Approximate methods for solving the traveling salesman problem include various versions of the greedy algorithm, in which at each step the transition of the lowest cost is selected from a set of transitions from one point to another that do not violate the correctness of the solution, as well as versions of the k -opt local search algorithm, $k = 2, 3, \dots, K$. 2-opt, widely used in many combinatorial optimization problems, amounts to removing two transitions from a tour and inserting two new transitions with the condition that the solution is cyclic [7, 10]. A tour is considered a local minimum if it does not contain any pair of transitions whose replacement would improve the solution; k -opt is similar to 2-opt, but at each step k transitions are removed instead of two. k -opt algorithms with small k can be effective in solving traveling salesman problems for which the corresponding assignment problem has an optimal solution with k subcycles.

Another group of approximate metaheuristic algorithms (Genetic Algorithm, Ant Colony Optimization, Simulated Annealing) emerged from observations of special natural phenomena [6, 8]. They are characterized by repeated application of truth-like rules to achieve an acceptable approximate solution.

This article considers the traveling salesman problem, which has been intensively studied and widely used to solve frequently encountered practical problems. The main solution methods are analyzed theoretically and through computational experiments. Exact methods have exponential complexity and are therefore not always suitable for large-scale problems. Approximation algorithms are often used to obtain near-optimal solutions in a reasonable time. Methods for solving the traveling salesman problem are being refined, and combined algorithms are being developed. The traveling salesman problem is a transportation-type problem. This led to the idea of applying a natural method to solve the traveling salesman problem, using a modification of the Hungarian algorithm, which is very effective for solving the assignment problem. The novelty of this article lies in the development of a new algorithm that significantly exploits the properties of the Hungarian algorithm to obtain a cyclic optimal solution. Computational experiments have confirmed this as a useful tool for real-world use in solving the traveling salesman problem.

Keywords: *traveling salesman problem, cyclically independent elements, modified Hungarian method.*

Based on the above-mentioned methods for solving the traveling salesman problem, many different variations, improvements, and combined algorithms have been constructed [10, 12].

Computational experiments [8–14] revealed the following features of the methods: a) for many methods (including the branch and bound method) it is possible to construct such a problem (maybe even of small size), that obtaining an exact solution will be difficult [1]; b) in the search for a solution (when some operations related to probabilistic calculations are used) at the initial iterations there is a rapid decrease in the objective function, then a situation arises when with a large number of successive iterations the value of the objective function stabilizes.

As a result of the analysis of many results of computational experiments questions and problems arise about the choice of a method for solving the traveling salesman problem, about the need for a deeper understanding of existing approaches, new ideas.

Based on the hypothesis that for the traveling salesman problem, as a transport-type problem, it is possible to construct an exact polynomial algorithm, a natural approach is developed here based on the application of a modification of the Hungarian method for solving the assignment problem [15–18].

The novelty of this method lies in its application of the Hungarian algorithm's essentially useful property of introducing an independent zero into the solution at each iteration. When solving the traveling salesman problem, cyclically independent zeros are introduced into the solution, the number of which may be less than the dimension of the original cost matrix.

The idea of modifying the Hungarian algorithm to solve the traveling salesman problem is as follows.

Let C be a cost matrix of order n containing only non-negative elements, with the diagonal elements being sufficiently large numbers.

a) The matrix C is reduced by subtracting the minimum value for each row from each element in that row; similarly, for each column, find the minimum element and subtract it from each element in that column. This yields the reduced matrix D , in which each row and each column contains a zero element. The given matrix D can also be obtained as a result of solving the assignment problem with matrix C . In matrix D , as in the case of reduction, there will be at least one zero in each row and each column.

Definition. A set of independent cyclic elements (x_i, y_i) , $i = \overline{1, k}$, is a set of elements of the matrix D , $D(x_i, y_i)$, $i = \overline{1, k}$, that form part of a cyclic substitution. An element (a, b) is called an independent cyclic element if after adding it to the set of independent cyclic elements we obtain a new set of independent cyclic elements, otherwise the element (a, b) is called dependent.

b) We select the maximum possible set of independent cyclic zeros from the set of all zeros D .

We form two matrices D_1 and D_2 . Matrix D_1 consists of elements of matrix D at the intersection of rows x_1 and columns y_1 , containing the selected zeros. Matrix D_2 includes elements of matrix D at the intersection of rows x_2 and columns y_2 , not containing the selected zeros.

c) Introducing an additional independent cyclic zero. We find the minimum positive independent cyclic element mnz and simply the minimum positive element mp of the matrix D_2 .

If mp is less than mnz , then we form two matrices D_3 and D_4 from the elements of D_2 . The matrix D_3 consists of the elements of the matrix D_2 at the intersection of rows x_3 from the set x_2 and columns y_3 from y_2 , containing elements that are less than D_4 . The matrix D_4 consists of the elements of D_2 at the intersection of rows x_4 from x_2 and columns y_4 from y_2 , containing elements not less than mnz .

d) The main transformation of the matrix D is carried out depending on the values of mnz and mp . In the case of $mnz=mp$, we subtract mnz from each element of D_2 , and add mnz to the elements of D_1 . In the case of $mnz>mp$, we subtract mnz from each element of D included in D_4 , and add mnz to the elements of D at the intersection of rows with numbers from x_1 and x_3 and columns with numbers from y_1 and y_3 .

The main transformation of the matrix D at each iteration of the algorithm is equivalent to constructing an equivalent matrix $\{D_{ij} + p_i + q_j\}$, $i, j = \overline{1, n}$, where p_i and q_j are specially selected numbers.

An algorithm for solving the traveling salesman problem using a modification of the Hungarian method.

As in the case of solving the assignment problem using the Hungarian method, the possibility of obtaining additional independent zeros for solving the traveling salesman problem is possible using equivalent transformations of the matrix C equivalent to D ,

$$D_{ij} = C_{ij} + p_i + q_j, \quad i, j = \overline{1, n}, \quad (1)$$

where p_i, q_j are arbitrary numbers. Equivalent transformations of type (1) lead to coinciding optimal assignments of assignment problems with matrices C and D . A similar situation occurs in the traveling salesman problem.

The theoretical part and computational scheme for solving the traveling salesman problem are reduced to constructing vectors p and q in order to increase the number of independent cyclic zeros from iteration to iteration.

1. Formation of the traveling salesman problem for a minimum with the cost matrix $\{C_{ij}\}$, $i, j = \overline{1, n}$.

We consider the elements of C_{ij} to be non-negative numbers, $C_{ij} = Inf$ for $i = j$, where Inf is a sufficiently large number.

2. The problem of choice with matrix C is solved. We obtain a solution in the form of a substitution (i_s, j_s) , $s = \overline{1, n}$, and a reduced matrix $D = \{D_{ij}\}$, $i, j = \overline{1, n}$, in which each row and each column contains at least one zero, $D_{i_s, j_s} = 0$, $s = \overline{1, n}$.

3. If the solution to the (i_s, j_s) is a cyclic permutation, then the traveling salesman problem is solved.

4. Start of iteration t .

5. We generate a cyclic solution (xv, yv) with the value fv from a non-decreasing sequence of elements of the matrix D , selecting from all its zero elements as many independent cyclic zeros (x_{1i}, y_{1i}) , $D(x_{1i}, y_{1i}) = 0$, $i, j = \overline{1, h_1}$ as possible.

6. We select the best solution based on the value of fv :

$$f_{opt} = fv, \quad x_{opt} = xv, \quad y_{opt} = yv.$$

If the number of independent cyclic zeros satisfies the condition $h_1 \geq n - 1$, then, due to the cyclicity of (xv, yv) , the optimality of the solution (x_{opt}, y_{opt}) will follow.

7. Form two matrices D_1 and D_2 . Matrix D_1 consists of elements of matrix D located at the intersection of rows x_{1i} and columns y_{1i} , $i, j = \overline{1, h_1}$. Matrix D_2 consists of elements of matrix D at the intersection of rows x_{2i} and columns y_{2i} , $i, j = \overline{1, h_2}$, $h_2 = n - h_1$,

$$x_2 = \{1, 2, \dots, n\} \setminus x_1, \quad y_2 = \{1, 2, \dots, n\} \setminus y_1.$$

In vector form $D_1 = D_1(x_1, y_1)$, $D_2 = D_2(x_2, y_2)$. We will say that matrix D_1 consists of occupied elements, and D_2 of unoccupied elements.

8. Determine the minimum cyclic independent element of the matrix D_2 ,

$$mnz = \min_{1 \leq i, j \leq n} D_2(i, j) = D_2(r, s).$$

Note that if we introduce variables $rt = x_2(r)$ and $st = y_2(s)$, then we can write $mnz = D(rt, st)$.

9. We determine the following values of the features nn, ne, zn and ze for mnz :

$$nn = \begin{cases} 1, & \text{if } mnz = 0 \\ 0, & \text{otherwise;} \end{cases}$$

$$ne = \begin{cases} 1, & \text{if } mnz > 0 \\ 0, & \text{otherwise;} \end{cases}$$

$$zn = \begin{cases} 1, & \text{if matrix } D_2 \text{ has a dependent} \\ & \text{zero and } mnz > 0; \\ 0, & \text{otherwise} \end{cases}$$

$$ze = \begin{cases} 1, & \text{if matrix } D_2 \text{ contains a dependent} \\ & \text{element } a > 0, \text{ but smaller than } mnz. \\ 0, & \text{otherwise} \end{cases}$$

10. Depending on the value of mnz and the values of its features, the method of transforming the matrix D and adding independent cyclic zeros to the set (x_{1i}, y_{1i}) , $i, j = \overline{1, h_1}$, is determined.

There can be three different cases:

- 1) $nn = 1$;
- 2) $ne = 1$ and at least one of the features zn or ze is equal to one;
- 3) $ne = 1$, while zn and ze are equal to zero.

11. In the first case, if $nn = 1$, then matrix D does not change, but matrices D_1 and D_2 change. Matrix D_1 expands and consists of elements of matrix D at the intersection of rows from x_1 and rt and columns from y_1 and st , and D_2 contracts and consists of elements of D at the intersection of rows $x_2 \setminus rt$ and columns $y_2 \setminus st$.

12. Select all independent cyclic zeros from the matrix D_2 , if there are any, repeating steps 8–11.

If the set (x_1, y_1) contains at least $n - 1$ independent cyclic zeros, then an optimal solution is formed.

13. In the second case, if $ne = 1$ and $zn = 1$ or $ze = 1$, then an independent cyclic positive element is introduced into the set (x_1, y_1) . To do this, all dependent zeros and positive elements smaller than the minimum independent cyclic positive element mnz are crossed out with a minimum number of lines (rows and columns). The numbers of the crossed out rows x_3 are entered into the set x_1 , and the numbers of the crossed out columns y_3 are entered into the set y_1 . Thus, we obtain the updated x_1 and y_1 . Now we construct the matrices D_1 and D_2 . At the intersection of the rows with numbers from x_1 and the columns with numbers from y_1 of the matrix D , we obtain the matrix D_1 , and D_2 includes all the elements from D at the

intersection of the remaining row numbers x_2 after removing x_1 from $N = \{1, 2, \dots, n\}$ and the columns y_2 after removing y_1 from the set N :

$$x_2 = N \setminus x_1, \quad y_2 = N \setminus y_1.$$

14. We carry out the basic transformation of matrix D : we subtract mnz , $D(x_2, y_2) = D(x_2, y_2) - mnz$, from all elements of matrix D included in D_2 , and add mnz , $D(x_1, y_1) = D(x_1, y_1) + mnz$, to all elements of matrix D included in D_1 . The remaining elements of matrix B remain unchanged.

15. In the third case, when $ne = 1$ together with $zn = 0$ and $ze = 0$, only the main transformation of matrix D is carried out as in item 14, without preliminary transformations of matrices D_1 and D_2 as in item 13, i.e. mnz is subtracted from the elements of $D(x_2, y_2)$, and mnz is added to the elements of $D(x_1, y_1)$. The remaining elements of D do not change.

16. After the main transformation of the matrix D , as in paragraphs 14 and 15, its reduction is performed.

17. The iteration t of the algorithm ends with the calculation of the estimate of the value of the current solution:

$$R_{t+1} = R_t + (n - k) * mnz + sm,$$

where mnz is the minimum cyclic independent element selected from the matrix D_2 , k is the number of selected independent cyclic zeros at the current iteration, sm is the sum of the minimum elements of the rows and columns after the reduction of the matrix D .

18. End of iteration t . The finiteness of the algorithm is established by constructing a matrix D equivalent to the matrix C , with n independent zeros or for $R_t \geq \text{fopt}$, where fopt is the value of the best cyclic solution.

Computational experiments. The traveling salesman problems with a cost matrix of order n from 10 to 100 and of varying complexity, i.e. different numbers of subcycles in the optimal solution of the corresponding assignment problem, were solved. The results obtained for one of the implementations of the algorithm are presented in the Table, where i is the average number of iterations, t is the average time to solve the problem.

TABLE. The results for one of the implementations

n	10	20	30	40	50	60	70	80	90	100
i	3	5	8	19	28	30	33	40	52	66
$t, \text{ sec.}$	0,2	0,4	1,0	4,3	11,2	18,8	37,2	80,7	146,5	320,9

Computational experiments have shown that the constructed algorithm allows obtaining exact solutions and has the property of approximate algorithms, allowing finding good solutions for a sufficiently large n in a relatively short time, linearly proportional to the dimension and the value of the maximum element of the cost matrix, i.e. it is a useful tool that expands the possibilities of solving the traveling salesman problem.

Conclusion. An analysis of the results of numerous computational experiments on methods for solving the traveling salesman problem shows that new ideas and further improvement of existing approaches are required for its stable and accurate solution. In this sense, the idea of modifying the Hungarian method fits within the framework of a natural method for solving the traveling salesman problem as a transportation-type problem. A new exact algorithm for solving the traveling salesman problem has been developed, based

on the useful properties of the Hungarian algorithm for efficiently solving the assignment problem. Computational experiments on various implementations of this algorithm have been conducted, confirming a number of computational advantages over the branch-and-bound method: the solution requires only input information, requiring virtually no additional memory; the number of iterations to complete the computational process is in many cases almost proportional to the dimension of the input matrix of the problem.

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Received 01.08.2025

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Вступ. Задача комівояжера – важлива модель для проведення досліджень у різних галузях науки, економіки та техніки. Побудова ефективних алгоритмів для пошуку оптимальних розв'язків – актуальне завдання. Задача комівояжера це задача транспортного типу і природнім підходом до її розв'язування можуть бути алгоритми, що ґрунтуються на методах розв'язування транспортних задач, зокрема, варіанти угорського методу.

Мета. Розробка модифікації угорського методу для розв'язування задачі комівояжера. Розширення можливостей точних методів розв'язування задач транспортного типу.

Результати. Показана можливість розв'язування задачі комівояжера модифікацією угорського методу для задачі призначення. Введено поняття набору циклічно незалежних елементів та циклічно незалежного нуля даної матриці. У цих нових термінах викладено ідею та одну з її реалізацій. Було проведено обчислювальні експерименти. Результати відображаються трьома показниками: середньою кількістю ітерацій та процесорним часом для набору задач заданої розмірності.

Висновки. Потреба розробки ефективних точних методів розв'язування задачі комівояжера призводить до появи нових ідей, підходів. У цьому сенсі корисним є застосування природнього підходу (заснованого на застосуванні методів розв'язування задач транспортного типу), оскільки задача комівояжера є задачею транспортного типу. Показано, що ґрунтуючись на модифікації угорського методу можна будувати ефективні методи отримання точного чи наближеного розв'язку залежно від конкретної ситуації.

Ключові слова: задача комівояжера, циклічно незалежні елементи, модифікація угорського методу.